Ma 145a: Homework set 1, Due October 10 at noon

We assume the base field K is always algebraically closed and we assume the Schur's lemma.

1. Let (ρ, V) be a representation of G of degree n. Assume $\det \rho(g) = 1$ for all $g \in G$. Then for $0 \leq k \leq n$, the representation $\wedge^k V$ is isomorphic to $\wedge^{n-k} V^*$.

2. Let (ρ, V) be an irreducible representation of degree n of a finite group G. Let χ_V be its character. Assume Z is the center of G. Let g and c be the order of G and Z respectively.

   (a) Show that Z acts on V by scalars via $\rho|_Z$. Deduce from it that $|\chi_V(z)| = n$.

   (b) Prove that $n^2 \leq g/c$. (We have seen $n \leq g/c$.)

   (c) Show that if ρ is faithful (i.e. $\rho : G \to \mathrm{GL}(V)$ is injective) then Z is a cyclic group.

3. Let H be a subgroup of a finite group G. Show that each irreducible representation of G is contained in a representation induced by an irreducible representation of H.

4. Assume K has positive characteristic p and G has order g. Show that the following two properties are equivalent:

   (a) The group algebra $K[G]$ is semisimple. (Equivalently, every $K[G]$-module is semisimple.)

   (b) The characteristic p of K does not divide the order g of G.