Short Proofs of Classical Theorems

J. A. Bondy

UNIVERSITÉ CLAUDE–BERNARD LYON 1 AND UNIVERSITÉ PARIS 6

E-mail: jabondy@moka.ccr.jussieu.fr

Received January 10, 2002; Revised May 1, 2003

DOI 10.1002/jgt.10135

Abstract: We give proofs of Ore's theorem on Hamilton circuits, Brooks' theorem on vertex coloring, and Vizing's theorem on edge coloring, as well as the Chvátal-Lovász theorem on semi-kernels, a theorem of Lu on spanning arborescences of tournaments, and a theorem of Gutin on diameters of orientations of graphs. These proofs, while not radically different from existing ones, are perhaps simpler and more natural.
J Graph Theory 44: 159–165, 2003

Keywords: *Brooks' theorem; Vizing's theorem; digraphs; semi-kernels; diameter; depth-first-search*

1. ORE'S THEOREM

Our proof of Ore's theorem [15] bears a close resemblance to the proof of Dirac's theorem [5] given by Newman [14], but is more direct.

Ore's Theorem. *Let G be a simple graph on at least three vertices, in which the sum of the degrees of any two non-adjacent vertices is at least $v(G)$. Then G contains a Hamilton circuit.*

Proof. Consider the complete graph K on the vertex set V of G in which the edges of G are colored blue and the remaining edges of K are colored red. Let C be a Hamilton circuit of K with as many blue edges as possible. We show that every edge of C is blue, in other words, that C is a Hamilton circuit of G.

Suppose, to the contrary, that C has a red edge xx^+ (where x^+ denotes the successor of x on C). Consider the set S of vertices joined to x by blue edges (that is, the set of neighbors of x in G). The successor x^+ of x on C must be joined by a blue edge to some vertex y^+ of S^+, because if x^+ is adjacent in G only to vertices in $V\backslash(S^+ \cup \{x^+\})$,

$$d_G(x) + d_G(x^+) = |N_G(x)| + |N_G(x^+)| \leq |S| + (|V| - |S^+| - 1) = v(G) - 1,$$

contradicting the hypothesis that $d_G(x) + d_G(x^+) \geq v(G)$, x and x^+ being non-adjacent in G. But now the circuit C' obtained from C by exchanging the edges xx^+ and yy^+ for the edges xy and x^+y^+ has more blue edges than C, a contradiction. ■

Remark. The above proof shows, in effect, that Flood's procedure 2-*opt* for finding a relatively short traveling-salesman tour in a weighted complete graph [7] is guaranteed to return an optimal tour when the weight function is $(0, 1)$ and the spanning subgraph G induced by the edges of weight 0 satisfies the hypotheses of Ore's theorem.

2. BROOKS' THEOREM

Our proof of Brooks' theorem [2], while similar in spirit to the one given by Lovász [11], makes essential use of depth-first-search trees. It also appeals to the characterization by Chartrand and Kronk [3] of those graphs in which every such tree is a Hamilton path. For completeness, we include a proof of this latter result.

Proposition. *Let G be a connected graph every depth-first-search tree of which is a Hamilton path. Then G is a circuit, a complete graph, or a complete bipartite graph $K_{n,n}$.*

Proof. Let P be a Hamilton path of G, with origin x. Because the path $P - x$ extends to a Hamilton path of G, the path P extends to a Hamilton circuit C of G. If C has no chord, $G = C$ is a circuit. So let xy be a chord of C. Then x^+y^+ is one too, because $x^+CyxC^{-1}y^+$ is a Hamilton path of G; likewise, x^-y^- is a chord of C. And if the length of xCy is at least four, $x^{++}y$ and x^+y^- are also chords of C, in view of the Hamilton path $x^{++}Cy^-x^-C^{-1}y^+x^+xy$ and the fact that $x^+y^- = (x^{++})^-y^-$.

If C has a chord xz of length two, let $y := x^+(= z^-)$. Then $yz^+ \in E$. Moreover, if $yz^{+i} \in E$, then $yz^{+(i+1)} \in E$ in view of the Hamilton path $z^{+(i+1)}CxzCz^{+i}y$. It follows that y is adjacent to every vertex of G. But then G is complete, because

$x^{+i}z^{+i}$ is a chord of length two for all i. If C has no chord of length two, every chord of C is odd; moreover, every odd chord must be present. Thus $G = K_{n,n}$, where $v(G) = 2n$. ■

Brooks' Theorem. *If G is a connected graph, and is neither an odd circuit nor a complete graph,* $\chi \le \Delta$.

Proof. Suppose first that G is not regular. Let x be a vertex of degree δ and let T be a search tree of G rooted at x. We color the vertices with the colors $1, 2, \ldots, \Delta$ according to the greedy heuristic, selecting at each step a leaf of the subtree of T induced by the vertices not yet colored, assigning to it the smallest available color, and ending with the root x of T. When each vertex v different from x is colored, it is adjacent (in T) to at least one uncolored vertex, and so is adjacent to at most $d(v) - 1 \le \Delta - 1$ colored vertices. It is therefore assigned one of the colors $1, 2, \ldots, \Delta$. Finally, when x is colored, it, too, is assigned one of the colors $1, 2, \ldots, \Delta$, because $d(x) = \delta \le \Delta - 1$. The greedy heuristic therefore produces a Δ-coloring of G.

Suppose now that G is regular. If G has a cut vertex x, then $G = G_1 \cup G_2$, where G_1 and G_2 are connected and $G_1 \cap G_2 = \{x\}$. Because the degree of x in G_i is less than $\Delta(G)$, neither subgraph G_i is regular, so $\chi(G_i) \le \Delta(G_i) = \Delta(G)$, $i = 1, 2$, and $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} \le \Delta(G)$. We may assume, therefore, that G is 2-connected.

If every depth-first-search tree of G is a Hamilton path, G is a circuit, a complete graph, or a complete bipartite graph $K_{n,n}$, by the proposition. Since, by hypothesis, G is neither an odd circuit nor a complete graph, $\chi(G) = 2 \le \Delta(G)$. Suppose, then, that T is a depth-first-search tree of G, but not a path. Let x be a vertex of T with at least two children, y and z. Because G is 2-connected, both $G - y$ and $G - z$ are connected. Thus there are proper descendants of y and z, each of which is joined to an ancestor of x, and it follows that $G' := G - \{y, z\}$ is connected. Consider a search tree T' with root x in G'. By coloring y and z with color 1, and then the vertices of T' by the greedy heuristic as above, ending with the root x, we obtain a Δ-coloring of G. ■

3. VIZING'S THEOREM

Vizing's Theorem. *For any simple graph G,* $\chi' \le \Delta + 1$.

This theorem was found independently by Vizing [16] and Gupta [9]. Following the approach of Ehrenfeucht, Faber, and Kierstead [6], we prove the theorem by induction, assuming that there is a $(\Delta + 1)$-edge coloring of $G - v$, where $v \in V$. To complete the proof, it suffices to show how a $(\Delta + 1)$-edge coloring of G itself can be obtained from this $(\Delta + 1)$-edge coloring of $G - v$. We show that this leads naturally to a bipartite matching problem. Our proof can be adapted without difficulty to establish the multigraph version of Vizing's theorem. We say that color α is *represented* at vertex v if it is assigned to some edge incident with v; otherwise it is *available* at v.

Lemma. *Let G be a simple graph, let v be a vertex of G, and let $k \geq \Delta$ be an integer. Suppose that $G - v$ has a k-edge coloring with respect to which every neighbor of v has at least two available colors, except possibly one vertex, which has at least one available color. Then G is k-edge-colorable.*

Remark. The hypothesis of the lemma is satisfied if $k = \Delta + 1$, because each neighbor of v has degree at most $\Delta - 1$ in $G - v$. Vizing's theorem thus follows directly by induction on $v(G)$. The lemma also implies the theorem of Fournier [8], that a simple graph G is Δ-edge-colorable if its vertices of degree Δ induce a forest F. To see this, it suffices to choose v to be a vertex of degree at most one in F and apply induction, with $k = \Delta$.

Proof of Lemma. Set $X := N(v)$, and let c be a k-edge coloring of $G - v$. To complete c to a k-edge coloring of G, we must find a system of distinct representatives of the family $\mathcal{A} := (A(x) : x \in X)$ consisting of the sets of colors available at the neighbors of v.

Denote by Y the set of k colors available to color the edges of G. We form a bipartite graph H with bipartition (X, Y), where $x \in X$ is adjacent to $y \in Y$ if and only if $y \in A(x)$. The question of whether the family $\mathcal{A}$ has an SDR is then equivalent to that of whether the graph H has a matching saturating X. The sets $A(x)$, and hence the graph H, depend on the k-edge coloring c of $G - v$. Our task is to show that there exists a coloring c such that H does indeed have a matching saturating X.

Suppose, to the contrary, that c has been chosen so that H has a matching M which is as large as possible, but that M does not saturate X. If xy is an edge of M, we assign the color y to the edge xv of G; the matching M, therefore, defines an extension $\tilde{c}$ of the coloring c to a proper subset of the set of edges incident to v. Let u be an M-unsaturated vertex of X; thus u is a neighbor of v in G for which the edge uv remains uncolored. Denote by S and T, respectively, the sets of vertices of X and Y which are reachable from u by M-alternating paths in H. Because M is a maximum matching, each vertex of T is matched with a vertex of $S \backslash u$. It follows that $|S \backslash u| = |T|$ and that $A(x) \subseteq T$ for all $x \in S$.

We shall now show how the coloring c of $G - v$ may be modified to a coloring c' of $G - v$ so that the associated bipartite graph H' has a matching M' with $|M'| = |M| + 1$. This contradiction to the choice of c will establish the theorem.

Consider the coloring $\tilde{c}$. For $x \in S \cup \{v\}$, let $\tilde{A}(x)$ denote the set of colors available at x with respect to this coloring. Then $\tilde{A}(v) \subseteq Y \backslash T$, $\tilde{A}(u) = A(u)$ because the edge uv is still uncolored, and $\tilde{A}(x) = A(x) \backslash y$, for $x \in S \backslash u$, where y is the color assigned to the edge xv in $\tilde{c}$. In particular, $\tilde{A}(x) \subseteq A(x)$ for all $x \in S$, so $\cup_{x \in S} \tilde{A}(x) \subseteq T$. Because each vertex of $S \backslash u$ is M-saturated,

$$\sum_{x \in S} |\tilde{A}(x)| = \sum_{x \in S \backslash u} (|A(x)| - 1) + |A(u)| = \sum_{x \in S} |A(x)| - |S \backslash u| > 2|T| - |T| = |T|.$$

It follows that some color $\alpha \in T$ is still available at two vertices x, x' of S, but is represented at v because $\alpha \in T$. On the other hand, since $|M| < |X| = d(v) \leq \Delta \leq |Y|$, there is a color $\beta \in Y \backslash T$ which is still available at v, but which is represented at each vertex x of S because $\cup_{x \in S} \tilde{A}(x) \subseteq T$.

Consider now the $\alpha\beta$-subgraph of G. The vertices v, x, and x' all have degree one in this subgraph, and are, therefore, ends of $\alpha\beta$-paths. Clearly, at most one of the vertices x, x' can be the other end of the path starting at v. Suppose then that the path starting at v does not terminate at x', and let x'' be the terminus of the path starting at x'. Interchanging α and β on this path, we obtain a new coloring c' of $G - v$ with respect to which the color β is now available at x'. Let H' be the bipartite graph corresponding to c'. Then H' is identical to H, except that the edge $x'\alpha$ has been replaced by $x'\beta$, and the edge $x''\alpha$ has been replaced by $x''\beta$, or conversely, if $x'' \in X$. In particular, M is a matching in H', and u is connected to β by an M-augmenting path in H'. The matching M may thus be augmented to a matching M' in H' of cardinality $|M| + 1$. ■

4. THE CHVÁTAL-LOVÁSZ THEOREM

A *kernel* of a digraph G is a stable set S such that every vertex not in S dominates some vertex of S. A *semi-kernel* is a stable set S such that every vertex not in S either dominates some vertex of S or dominates a vertex which in turn dominates some vertex of S. Chvátal and Lovász [4] proved that every digraph has a semi-kernel. Our proof is based on the fact that every acyclic digraph has a (unique) kernel.

The Chvátal-Lovász Theorem. *Every digraph has a semi-kernel.*

Proof. Let G be a digraph, H a maximal induced acyclic subgraph of G, and S the kernel of H. We claim that S is a semi-kernel of G. Since S is a kernel of H, every vertex of $H - S$ dominates some vertex of S. Consider any vertex $v \in V(G) \backslash V(H)$. By the choice of H, there is a directed circuit C in the subgraph of G induced by $V(H) \cup \{v\}$. The vertex v thus dominates its successor v^+ on C. Because $v^+ \in V(H)$, either $v^+ \in S$ or v^+ dominates some vertex of S. ■

Remark. S. Thomassé (*personal communication*) has given an elegant variant of this proof. Consider an arbitrary total order $\prec$ of $V(G)$. Let G' and G'' be the spanning acyclic subgraphs of G induced by $\{(x, y) : x \prec y\}$ and $\{(x, y) : y \prec x\}$, respectively. Let S' be the kernel of G'. The kernel of $G''[S']$ is a semi-kernel of G.

5. LU'S THEOREM

A 2-*arborescence* is an arborescence in which each vertex other than the root has outdegree at most two. Studying unavoidable subgraphs of tournaments, Lu [12] established the following theorem, a refinement of the simple fact that every tournament has a semi-kernel (a *king* in the language of tournaments). A shorter proof was given by Lu, Wang, and Wong [13].

Lu's Theorem. *In any tournament, each vertex of maximum outdegree is the root of a spanning* 2*-arborescence of depth at most two.*

Proof. Let T be a tournament and v a vertex of maximum outdegree in T. Set $Y := N^+(v)$ and $X := V \backslash (Y \cup \{v\})$, and denote by $B(X, Y)$ the bipartite graph in which $x \in X$ is adjacent to $y \in Y$ if and only if y dominates x in T. For $S \subseteq X$, denote by $N(S)$ the set of neighbors of S in B. By a standard variant of Hall's theorem, T has a spanning 2-arborescence of depth at most two rooted at v if and only if, $|N(S)| \geq \frac{1}{2}|S|$, for all $S \subseteq X$. We verify this condition.

Let $S \subseteq X$. Some vertex $x \in S$ dominates at least $\frac{1}{2}(|S|-1)$ vertices of S. Moreover, x dominates v. On the other hand, by the choice of v, x dominates no more than $d^+(v) = |Y|$ vertices in all. It follows that x dominates at most $|Y| - \frac{1}{2}(|S|+1)$ vertices of Y, and thus is dominated by at least $\frac{1}{2}(|S|+1)$ vertices of Y. Therefore $|N(S)| \geq \frac{1}{2}|S|$. ■

6. GUTIN'S THEOREM

The *diameter* of a directed graph is the maximum distance $d(u, v)$ between two vertices u, v of the graph. Thus, a directed graph has finite diameter if and only if it is strong, hence an orientation of a 2-edge-connected graph.

Consider now a 2-edge-connected graph G. One may ask how large the diameter of a strong orientation of G can be. Gutin [10] (see also [1]) gave the answer in the following theorem.

Gutin's Theorem. *Let G be a* 2*-edge-connected graph. The maximum diameter of a strong orientation of G is equal to the length of a longest path of G.*

Proof. It is clear that the diameter of a strong orientation of G cannot exceed the length of a longest path of G; we prove the reverse inequality. Let $P[x, y]$ be a longest path of G. Consider a depth-first search tree T of G, with root x, which begins with the path P. Orient the edges of T as an out-arborescence $\vec{T}$ with root x. For each remaining edge, one end is a descendant in $\vec{T}$ of the other; orient these edges from descendant to ancestor. The resulting directed graph $\vec{G}$ satisfies the requirements of the theorem.

For each edge (u, v) of $\vec{T}$, there is an edge (u', v') with u' a (not necessarily proper) descendant of v and v' a (not necessarily proper) ancestor of u, because G is 2-edge-connected. It follows that the root x is reachable from every vertex, and thus that $\vec{G}$ is strong. Moreover, P is the only (x, y)-path in $\vec{G}$, so the diameter of $\vec{G}$ is at least the length of P. ■

ACKNOWLEDGMENTS

We thank U. S. R. Murty and S. Thomassé for helpful discussions, and F. Bonomo for pertinent remarks on an earlier version.

REFERENCES

[1] J. Bang-Jensen and G. Gutin, Digraphs. Theory, algorithms, and applications, Springer Monographs in Mathematics, Springer-Verlag London, London, 2001, p. 62.

[2] R. L. Brooks, On coloring the nodes of a network, Proc Cambridge Philos Soc, 37 (1941), 194–197.

[3] G. Chartrand and H. V. Kronk, Randomly traceable graphs, SIAM J Appl Math 16 (1968), 696–700.

[4] V. Chvátal and L. Lovász, Every directed graph has a semi-kernel, in Hypergraph Seminar, C. Berge and D. K. Ray-Chaudhuri, editors, Springer-Verlag, New York, 1974, p. 175.

[5] G. A. Dirac, Some theorems on abstract graphs, Proc London Math Soc 2 (1952), 69–81.

[6] A. Ehrenfeucht, V. Faber, and H. A. Kierstead, A new method of proving theorems on chromatic index, Discrete Math, 52 (1984), 159–164.

[7] M. M. Flood, The traveling-salesman problem, Operations Res 4 (1956), 61–75.

[8] J.-C. Fournier, Colorations des arêtes d'un graphe, Colloque sur la Théorie des Graphes (Brussels, 1973), Cahiers Centre Études Recherche Opér 15 (1973), 311–314.

[9] R. P. Gupta, The chromatic index and the degree of a graph, Notices Amer Math Soc, 13 (1966), Abstract 66T-429.

[10] G. Gutin, Minimizing and maximizing the diameter in orientations of graphs, Graphs Combin 10 (1994), 225–230.

[11] L. Lovász, Three short proofs in graph theory, J Combin Theory Ser B 19 (1975), 111–113.

[12] X. Lu, On avoidable and unavoidable trees, J Graph Theory 22 (1996), 335–346.

[13] X. Lu, D. W. Wang, and C. K. Wong, On avoidable and unavoidable claws, Discrete Math, 184 (1998), 259–265.

[14] D. J. Newman, A problem in graph theory, Amer Math Monthly 65 (1958), 611.

[15] O. Ore, Note on Hamilton circuits, Amer Math Monthly 67 (1960), 55.

[16] V. G. Vizing, On an estimate of the chromatic class of a p-graph (Russian), Diskret Analiz 3 (1964), 25–30.