Homework 5 : Ma121a - Combinatorics

Homework is due on Thursday the 4th of December at 12:00. While collaboration is encouraged, you must write your own solutions.

1) (a) Show that the number of partitions of \( n \) into parts not divisible by \( d \) equal to the number of partitions in which no part occurs more than \( d - 1 \) times. (2 marks)
(b) Find a recurrence formula for the number of partitions, denoted \( a_n \), into (not necessarily distinct) powers of 2. (2 marks)

2) Show that the partition function admits the following representation

\[
\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{1-x^n} \right). \]

(2 marks)

3) Let \( F \) be the combinatorial structure of an alternating permutation. An alternating permutation, \( \pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \) for \( n \) odd, satisfies

\[ \pi(1) < \pi(2) > \pi(3) < \pi(4) > \ldots < \pi(n-1) > \pi(n). \]

Give a combinatorial interpretation of the structure associated with the derivative of an exponential generating function to show \( F(x) = \tan(x) \). (4 marks)

4) For species involving unlabelled objects, we may easily derive an implicit functional relation satisfied by the ordinary generating function.

(a) The ordinary generating function, \( A(x) \), for the number of unlabelled plane ternary rooted trees satisfies the functional relation

\[ A(x) = 1 + xA(x)^3. \]

Show that

\[ A(x) = \sum_{n=1}^{\infty} \frac{1}{3n+1} \binom{3n+1}{n} x^n. \] (2 marks)

(b) The ordinary generating function, \( B(x) \), for the number of unlabelled (2,3)-rooted trees satisfies the functional relation

\[ B(x) = x + B(x)^2 + B(x)^3. \]

Show that

\[ B(x) = x + \sum_{n=2}^{\infty} \frac{1}{n} \sum_{j \leq n-1} \binom{n-1+j}{j} \binom{j}{n-1-j} x^n. \] (3 marks)

5) Consider a Young tableau of shape \((n,n)\). Define a sequence \( a_k, 1 \leq k \leq 2n \) by \( a_k = i \) if \( k \) is in row \( i \) of the tableau, \( i = 1, 2 \). Use this to show that the number of Young tableaux of this shape is the Catalan number \( u_{n+1} \). (2 marks)

6) Let \( N = \{1, 2, 3, 4\} \) and \((N, N; L_1)\) and \((N, N; L_2)\) be two Latin squares. We call these Latin squares orthogonal if for each of the \( n^2 \) values \((L_1(i, j), L_2(i, j))\) are different. For example

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 3 & 4 & 2 \\
2 & 4 & 3 & 1 \\
3 & 1 & 2 & 4 \\
4 & 2 & 1 & 3
\end{pmatrix}
\]

Show that it is impossible to have more than \( n - 1 \) pairwise orthogonal \( n \times n \) Latin squares. (3 marks)