You must do problems (1) through (4). Problem (5) is for extra credit.

(1) Let $K/k$ be a field extension, and $x \in K$. Fix a $k$-derivation $D$ of $K$, by which we mean an element of $\text{Der}_k(K, K)$ (which is often abbreviated as $\text{Der}_k(K)$). Prove the following:

(a) $D^n: K \to K$ is a $k$-linear map for every integer $n \geq 0$.
(b) If for some $n > 0$, we have

(i) $(D + x)^n(1) = 0$, but $(D + x)^{n-1}(1) \neq 0$,

then

(ii) $\exists y \in \text{Ker}(D^n) - \text{Ker}(D^{n-1})$ s.t. $x = D(y)/y$.

(c) (ii) $\implies$ (i).

(When $x = D(y)/y$, we call $x$ the logarithmic derivative of $y$ (relative to $D$).)

(2) Let $R$ be a commutative ring and $f \in R$. Put $X_f = \text{Spec}(R_f) \subset X := \text{Spec}(R)$. Show the equivalence of the following:

(a) $f$ is nilpotent;
(b) $f \in \bigcap \mathfrak{p} \in X \mathfrak{p}$;
(c) $X_f$ is empty;
(d) $R_f = 0$.

(3) (Exactness of localization) Let $R$ be a commutative ring $R$ and $S$ a multiplicative set in $R$. Suppose we are given a short exact sequence

$$0 \to L \to M \to N \to 0$$

of $R$-modules, show that the associated sequence

$$0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$$

is also exact.

(4) Let $R$ be a commutative ring which is Artinian, i.e., satisfies the descending chain condition on ideals. Prove the following:

(a) The maximal ideal spectrum of $R$, denoted $\text{maxSpec}(R)$, is a finite set.
(b) For some integer $n$, the map $R \to \prod_{\mathfrak{m} \in \text{maxSpec}(R)}(R/\mathfrak{m}^n)$ is a bijection.
(c) For each $n \geq 1$, $R/\mathfrak{m}^n$ is a local ring, i.e., has a unique maximal ideal.
Let $A \to B$ be a homomorphism of commutative rings, making $B$ an $A$-algebra. Consider the diagonal homomorphism

$$\Delta : B \otimes_A B \to B, \quad b_1 \otimes b_2 \mapsto b_1 b_2,$$

with kernel $I$. We may view $B \otimes_A B$ as a $B$-module by $(b, b_1 \otimes b_2) \mapsto bb_1 \otimes b_2$.

(a) Show that the $B$-module structure on $B \otimes_A B$ induces one on $I/I^2$.

(b) Show that the map $d : B \to I/I^2$ defined by setting $db = 1 \otimes b - b \otimes 1$ is an $A$-derivation of $B$.

(c) Prove that $(I/I^2, d)$ satisfies the universal property: For every $B$-module $M$, and any $A$-derivation $D$ of $B$ with values in $M$, there is a unique homomorphism $f : I/I^2 \to M$ such that $D = f \circ d$.

(d) There is a natural isomorphism of $I/I^2$ with the module of (relative) differentials $\Omega^1_{B/A}$ as defined in class.

Here is the definition of the module of differentials from class: Let $B$ be generated (as an $A$-algebra) by $\{x_i \mid i \in I\}$ subject to relations $\{g_j((x_i)) = 0, \ j \in J\}$. Then $\Omega^1_{B/A}$ is the module generated by the symbols $\{dx_i \mid i \in I\}$ modulo the relations $\sum_i \frac{\partial B_i}{\partial x_i} dx_i = 0$, for all $j \in J$.)