You must do problems (1) through (4). Problem (5) is for extra credit.

(1) Let \( k \) be a field and \( E = k(X) \), where \( X \) is a variable, i.e., transcendental over \( k \). Let \( Y = f(X)/g(X) \) be an element of \( E \) not in \( k \), written in lowest terms as a quotient of polynomials \( f, g \) in \( k[X] \) having no common factor (which means \( f, g \) have no common non-constant factor). Prove that \( Y \) is transcendental over \( k \), and that \( [k(X) : k(Y)] = \deg(f/g) \), the degree of \( f/g \), defined to be \( \max(\deg(f), \deg(g)) \).

(2) Using (1), describe the group of automorphisms of \( k(X) \) over \( k \).

(3) Let \( k \) be a field, \( x \) transcendental over \( k \), and \( F \neq k \) a field such that \( k \subset F \subset k(x) \). Let \( f(X) = X^n + a_1 X^{n-1} + \cdots + a_n \) be the minimal polynomial of \( x \) over \( F \). Write \( a_i = b_i(x)/b_0(x) \), with (minimally chosen) \( b_i(X) \in k[X] \) (\( \forall i \)). Not all the \( a_i \) can be in \( k \) (since \( x \) is transcendental over \( k \)). Pick \( a_i \in F - k \) and call it \( y \). We may write \( y = g(x)/h(x) \in F - k \), with \( g, h \in k[X] \) having no common factor. Put \( m = \deg(g/h) \). We may choose \( i \) such that \( m \) is maximal.
   (a) Put \( U(X, Z) = b_0(Z)X^n + b_1(Z)X^{n-1} + \cdots + b_n(Z) \).
   Show that \( U(X, Z) \) is a polynomial in \( Z \) of degree \( \geq m \) which is not divisible by any non-constant polynomial in \( k[Z] \).
   (b) Show that \( g(X)h(Z) - g(Z)h(X) \) is a polynomial of degree \( \leq m \) in each of the variables \( X, Z \), which is not divisible by any non-constant polynomial in \( k[X] \), but is divisible by \( U(X, Z) \).

(4) Let \( k, F, x, y, m \) be as in (3). Making use of the results of (3), prove the following:
   (a) \( [k(x) : k(y)] = m \).
   (b) \( F = k(y) \).
   (The assertion of (b) is called Lüroth’s theorem.)

(5) Let \( k \) and \( E = k(x_1, \ldots, x_n, t) \) be fields, with \( x_1, x_2, \ldots, x_n \) algebraically independent over \( k \), and \( t \) algebraic over \( K := k(x_1, \ldots, x_n) \). Prove the following:
   (a) There is a unique (up to a multiple) polynomial \( f \in k[X_1, \ldots, X_n, T] \) such that \( f(x_1, \ldots, x_n, t) = 0 \) and \( f \) generates the ideal of all polynomials which vanish at \( (x_1, \ldots, x_n, t) \).
(b) $[E : K]$ is the degree of $f$ in $T$.
(c) $E/K$ is separable iff $\frac{\partial f}{\partial T} \neq 0$. 