You must do problems (1) through (4). Problem (5) is for extra credit.

(1)
(a) If \( f : A \to B \) is a homomorphism of commutative rings and \( P \) a prime ideal in \( B \), show that \( f^{-1}(P) := \{ a \in A \mid f(a) \in P \} \) is a prime ideal in \( A \).
(b) Give examples where \( P \) is maximal, but \( f^{-1}(P) \) is not, and vice versa.
(c) Suppose \( A = A_1 \times A_2 \) is a direct product of (commutative) rings, with projections \( f_j : A \to A_j, j = 1, 2 \). Prove that, for each prime ideal \( Q \) in \( A \), there is a unique \( j \in \{1, 2\} \) and a prime ideal \( P_j \) in \( A_j \) such that \( Q = f_j^{-1}(P_j) \).

(2) Let \( \varphi : A \to B \) be a homomorphism of commutative rings such that \( B \) is integral over \( A \). Prove the following:
(a) If \( \varphi \) is injective and \( B \) an integral domain, the \( B \) is a field iff \( A \) is a field.
(b) If \( Q \) is a prime ideal of \( B \) and \( P = \varphi^{-1}(Q) \), then \( Q \) is a maximal ideal iff \( P \) is one.
(c) If \( A \) is a local ring, i.e., has a unique maximal ideal \( M \), then there is a maximal ideal \( M' \) of \( B \) such that \( M = \varphi^{-1}(M') \).

(3) Let \( A \) be a Noetherian integral domain with fraction field \( F \), and \( E \) a finite separable extension of \( F \). Denote by \( B \) the ring of all the elements of \( E \) which are integral over \( A \), called the integral closure of \( A \) in \( E \). Suppose furthermore that \( A \) is integrally closed in \( F \), i.e., contains all the elements of \( F \) which are integral over \( A \). The show that \( B \) is finitely generated as an \( A \)-module.

(4) Let \( f_1, \ldots, f_r \) be polynomials in \( R := k[X_1, \ldots, X_n] \), with \( k \) an algebraically closed field, such that they have no common zero. Show that the ideal they generate is all of \( R \).
(Hint: Think about what Weak N"ullstellensatz says!)
Give a counterexample for a non-algebraically closed field \( k \).

(5) Let \( R \) be a Noetherian ring, \( B \) a finitely generated \( R \)-algebra, and \( G \) a finite group of \( R \)-algebra automorphisms of \( B \). Put \( A = B^G \), the subset of elements in \( B \) which are fixed by \( G \). Show that \( A \) is a finitely generated \( R \)-algebra.
(Hint: Given any \( \beta \in B \), exhibit a monic polynomial equation over \( A \) which it satisfies.)