1) Show that there is a partial recursive function \( f : \mathbb{N} \to \mathbb{N} \) for which there is no total recursive extension \( g \supseteq f \), i.e., there is no total recursive \( g : \mathbb{N} \to \mathbb{N} \) such that for any \( n \) in the domain of \( f \) we have \( f(n) = g(n) \).

\( \text{Hint.} \) Use diagonalization on a universal recursive function.

2) Show that \( R \subseteq \mathbb{N} \) is r.e. iff \( R \) is finite or there is a 1-1 recursive function \( f : \mathbb{N} \to \mathbb{N} \) with range(\( f \)) = \( R \).

3) Show that \( R \subseteq \mathbb{N} \) is recursive iff \( R \) is finite or there is a strictly increasing recursive total function \( f : \mathbb{N} \to \mathbb{N} \) with range(\( f \)) = \( R \).

4) Show that every infinite r.e. set \( R \subseteq \mathbb{N} \) contains an infinite recursive subset \( P \subseteq R \).

5) (The Selection Theorem for r.e. sets.) Show that if \( R \subseteq \mathbb{N}^2 \) is r.e., then there is a partial recursive function \( f : \mathbb{N} \to \mathbb{N} \) such that
   i) \( f(x) \downarrow \iff \exists y R(x, y) \),
   ii) \( \exists y R(x, y) \Rightarrow R(x, f(x)) \).

6* ) (The Reduction Theorem for r.e. sets.) Show that if \( A, B \subseteq \mathbb{N}^n \) are r.e., then there are r.e. sets \( A^*, B^* \subseteq \mathbb{N}^n \) such that

\[
A^* \subseteq A, \, B^* \subseteq B, \, A^* \cap B^* = \emptyset, \, A^* \cup B^* = A \cup B.
\]

7) (The Separation Theorem for co-r.e. sets.) A set is co-r.e. if its complement is r.e. Show that if \( A, B \subseteq \mathbb{N} \) are co-r.e. sets and \( A \cap B = \emptyset \), then there is a recursive set \( C \subseteq \mathbb{N} \) such that

\[
A \subseteq C, \, B \cap C = \emptyset.
\]

\( \text{Hint.} \) Use 6).

Note: Homework rules are posted in the Ma/CS 117a web page. The first five problems are worth 12 points each and the last two 20 points each.