 ASSIGNMENT #A

Due Tuesday, October 14 at 1:00 pm

1) Let $e$ be the basis of natural logarithms. Put

$$g(n) = \lfloor ne \rfloor = \text{the largest integer } \leq ne.$$ 

Show that $g: \mathbb{N} \rightarrow \mathbb{N}$ is primitive recursive.

[Hint: Show that there is no integer $x$ such that

$$n \sum_{k=0}^{n} \frac{1}{k!} < x < n \left( \sum_{k=0}^{n} \frac{1}{k!} + \frac{1}{n!} \right).$$

Conclude that $\lfloor ne \rfloor = \lfloor \sum_{k=0}^{n} \frac{n!}{k!} \rfloor$.]

Use this to show that $f(n) = a_n$, where $a_n = n$th digit in the decimal expansion of $e$, is primitive recursive.

2*) Let $H: \mathbb{N} \rightarrow \mathbb{N}$ be defined by:

$$H(n) = \begin{cases} 0, & \text{if there is a sequence of at least } n \\ 1, & \text{consecutive 7's in the decimal expansion of } \pi; \end{cases}$$

Is $H$ primitive recursive? Justify your answer.

3) Consider the following type of recursion (a form of recursion with substitution in the parameters)

$$f(0,n) = g(n),$$

$$f(m+1,n) = h(f(m,S(m,n))),$$

where $h, g, S$ are given functions on $\mathbb{N}$. Show that if $h, g, S$ are primitive recursive, so if $f$.

[Hint: Notice that for $m > 0$

$$f(m,n) = h(f(m-1,S(m-1,n)))$$

$$= h^{(2)}(f(m-2,S(m-2,S(m-1,n))))$$

$$\ldots$$

$$= h^{(m)}(g(S(0,S(1,S(2,\ldots S(m-2,S(m-1,n))\ldots)))))).$$]
Look at the numbers

\[ n, S(m - 1, n), S(m - 2, S(m - 1, n)), \ldots \]

which occur in building up this expression.]

4) Consider the following type of recursion (which is a special case of a form called unnested double recursion - the recursion used in defining Ackermann’s function is an example of a nested doubled recursion):

\[
\begin{align*}
    f(0, n) &= g(n) \\
    f(m + 1, 0) &= h(m) \\
    f(m + 1, n + 1) &= p(f(m + 1, n), f(m, n + 1), m, n).
\end{align*}
\]

Show that if \( g, h, p \) are primitive recursive, so is \( f \).

[Hint: Define the auxiliary function

\[ F(k) = \langle f(0, k), f(1, k - 1), \ldots, f(k, 0) \rangle. \]

Express \( f \) in terms of \( F \) and show that \( F \) is primitive recursive.]

5) Exercise (5) from Exercise Set #1.

6) Let \( \sqrt{2} = a_0.a_1a_2 \ldots \) be the decimal expansion of the square root of 2 (thus \( a_0 = 1, a_1 = 4, a_2 = 1, \ldots \)). Show that the function \( f : \mathbb{N} \to \mathbb{N} \) given by \( f(n) = a_n \) is primitive recursive.

Note: Homework rules are posted in the Ma/CS 117a web page. The first and the last four problems are worth 15 points each and the second (starred-no collaboration) problem 25 points.