PROBLEM SET NO. 7, PART II

• Let $X = \{(x_n)_{n \in \mathbb{N}} : \sum_{n \in \mathbb{N}} |x_n| < \infty\}$ and
  
  $\|x\| = \sum_{n \in \mathbb{N}} |x_n|$ for $x = (x_n)_{n \in \mathbb{N}}$.

  Show that $\| \cdot \|$ is a norm on $X$ and show that $X$ is complete with respect to the corresponding metric.

• Define $G : \mathbb{R}^2 \to \mathbb{R}$ by

  $G(x, y) := \begin{cases} 
  \frac{xy}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0), \\
  0, & \text{if } (x, y) = (0, 0). 
  \end{cases}$

  Show that $G$ is continuous, and all its partial derivatives exist on all of $\mathbb{R}^2$, but $G$ is not differentiable in the sense of Fréchet.

• Let $X, Y, Z$ be three normed spaces. Prove that if $F : X \to Y$ is differentiable at $x_0 \in X$, and $G : Y \to Z$ is differentiable at $y_0 = F(x_0) \in Y$, then $G \circ F$ is differentiable at $x_0$ and

  $$(G \circ F)'(x_0) = G'(F(x_0))F'(x_0).$$

Problem set due on Wednesday, 12/3, 4:00 PM, in math department’s drop box.