The 15 Puzzle

- **Problem.** Start with the configuration on the left and move the tiles to obtain the configuration on the right.
The 15 Puzzle (cont.)

- The game became a craze in the U.S. in 1880.
- **Sam Loyd**, a famous chess player and puzzle composer, offered a $1,000 prize for anyone who could provide a solution.

Reminder: Permutations

- **Problem.** Given a set \( \{1, 2, \ldots, n\} \), in how many ways can we order it?
- **The case** \( n = 3 \). Six distinct orders / permutations: 123, 132, 213, 231, 312, 321.
- **The general case.**
  \[
  n! = n \cdot (n - 1) \cdot \cdots \cdot 2 \cdot 1
  \]
The 15 Puzzle and Permutations

- How a configuration of the puzzle can be described as a permutation?
  - Denote the missing tile as 16.
  - The board below corresponds to the permutation:
    \[ 1 \ 16 \ 3 \ 4 \ 6 \ 2 \ 11 \ 10 \ 5 \ 8 \ 7 \ 9 \ 14 \ 12 \ 15 \ 13 \]

Permutations as Functions

- We can consider a permutation as a bijection from the set \( \{1,2,\ldots,n\} \) to itself.
  - Denote the bijection as \( \alpha \).
    - \( \alpha(1) = 5 \).
    - \( \alpha(3) = 3 \).
The Permutation Set $S_n$

- $S_n$ – The set of permutations of $\mathbb{N}_n = \{1,2,3,...,n\}$.
- We have $|S_n| = n!$.
- The set $S_3$:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
1 & 2 & 3 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
1 & 3 & 2 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
2 & 1 & 3 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
3 & 1 & 2 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
3 & 2 & 1 \\
\end{array}
\]

Combining Two Permutations

- $\alpha(1) = 2$, $\alpha(2) = 4$, $\alpha(3) = 5$, $\alpha(4) = 1$, $\alpha(5) = 3$.
- $\beta(1) = 3$, $\beta(2) = 5$, $\beta(3) = 1$, $\beta(4) = 4$, $\beta(5) = 2$.
- What is the function $\beta\alpha$.

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 4 & 5 & 1 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
5 & 4 & 2 & 3 & 1 \\
\end{array}
\]

First apply $\alpha$ and then $\beta$
Closure of $S_n$

- **Claim.** If $\alpha$ and $\beta$ are in $S_n$, so does $\alpha\beta$.
- By definition, $\alpha\beta$ is a function from $\mathbb{N}_n$ to itself.
- It remains to show that for every $i \in \mathbb{N}_n$ there is a unique $j \in \mathbb{N}_n$ such that $i = \alpha\beta(j)$.
  - Since $\alpha \in S_n$, there is a unique $k$ such that $i = \alpha(k)$.
  - Since $\beta \in S_n$, there is a unique $j$ such that $k = \beta(j)$.

Symmetry

- Is it true that for every $\alpha, \beta \in S_n$, we have $\alpha\beta = \beta\alpha$?
- No!
  - Consider $S_3$ and
    \[
    \begin{align*}
    \alpha &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} & \beta &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\
    \alpha\beta &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & \beta\alpha &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}
    \end{align*}
    \]
Associativity

- Is it true that for every $\alpha, \beta, \gamma \in S_n$, we have $(\alpha \beta)\gamma = \alpha(\beta \gamma)$?

- Yes. In both cases the product looks like:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$:</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>$\beta$:</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>$\gamma$:</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
</tr>
</tbody>
</table>

The Identity Element of $S_n$

- Identity. The identity permutation is defined as $\text{id}(r) = r$ for every $r \in \mathbb{N}_n$. For any $\alpha \in S_n$, we have $\text{id} \cdot \alpha = \alpha \cdot \text{id} = \alpha$. 

Identity crisis.
Inverse

- Is it true that for every $\alpha \in S_n$, there exists an inverse permutation $\alpha^{-1} \in S_n$ satisfying $\alpha \alpha^{-1} = \alpha^{-1} \alpha = \text{id}$.

- Yes:

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  2 & 4 & 5 & 1 & 3 \\
  \end{array}
  \]

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  1 & 2 & 3 & 4 & 5 \\
  \end{array}
  \]

Cycle Notation

- We can consider a permutation as a set of cycles.

  \[
  \begin{array}{cccccccc}
  1 & 2 & 3 & 4 & 5 & 6 \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  5 & 6 & 3 & 2 & 1 & 4 \\
  \end{array}
  \]

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 \\
  1 \rightarrow 5 & 2 \rightarrow 6 \rightarrow 4 & 3 \\
  \end{array}
  \]

- We write this permutation as $(1 \ 5)(2 \ 6 \ 4)(3)$. 
Converting to Cycle Notation

- $\alpha(1) = 2$, $\alpha(2) = 4$, $\alpha(3) = 5$, $\alpha(4) = 1$, $\alpha(5) = 3$.

- We start with 1 and construct its cycle: $1 \to 2 \to 4 \to 1$.

- We then choose a number that was not considered yet: $3 \to 5 \to 3$.

- We’ve dealt with every number of $\mathbb{N}_5$, so the cycle notation is $(1 \, 2 \, 4)(3 \, 5)$.

Counting Cycles

- **Problem.** How many distinct cycles of length $k$ exist in $S_n$?

- **Solution.**
  - There are $\binom{n}{k}$ ways of choosing $k$ elements for the cycle.
  - There are $k!$ ways to order this elements.
  - Each cycle has $k$ different representations.

\[
\binom{n}{k} \frac{1}{k!} \equiv \frac{n!}{k \cdot (n-k)!}.
\]
Card Shuffling

**Problem.** Cards numbered 1 to 12 are picked up in row order and re-dealt in column order:

1 2 3 1 5 9
4 5 6 2 6 10
7 8 9 3 7 11
10 11 12 4 8 12

How many times do we need to repeat this procedure until the cards return to their original positions?

Finding a Permutation

1 2 3 1 5 9
4 5 6 2 6 10
7 8 9 3 7 11
10 11 12 4 8 12

- A reshuffling corresponds to a permutation.
- For example, after each reshuffling 6 will move to the previous position of 5.
Solution

\[
\begin{array}{cccc}
1 & 2 & 3 & 1 & 5 & 9 \\
4 & 5 & 6 & 2 & 6 & 10 \\
7 & 8 & 9 & 3 & 7 & 11 \\
10 & 11 & 12 & 4 & 8 & 12
\end{array}
\]

- The cycle structure of the permutation: \( \alpha = (1)(2 \ 5 \ 6 \ 10 \ 4)(3 \ 9 \ 11 \ 8 \ 7)(12) \).
- Every cycle has length 1 or 5, so after five steps we return to the original position.

Classification of Permutations

- The type of a permutation of \( S_n \) is the number of cycles of each length in its cycle structure.
- Both \((1 \ 2 \ 4)(3 \ 5)\) and \((1 \ 2 \ 3)(4 \ 5)\) are of the same type: one cycle of length 3 and one of length 2.
  - We denote this type as \([2 \ 3]\).
- In general, we write a type as \([1^{\alpha_1} \ 2^{\alpha_2} \ 3^{\alpha_3} \ 4^{\alpha_4} \ ...]\).
Counting Permutations of a Given Type

- **Problem.** How many permutations of $S_{14}$ are of the type $[2^2 3^2 4]$?

- We need to insert the numbers 1, 2, ..., 14 into the cycle pattern
  \[(\cdots)(\cdots)(\cdots)(\cdots)\cdots\].

- We can place every permutation of $\mathbb{N}_{14}$ into this pattern.
  - $(12 1)(3 5)(2 6 4)(13 14 3)(7 8 9 10)$
  - Is the solution $14!$?

Fixing the Solution

- The following permutations are identical:
  - $(12 1)(3 5)(2 6 4)(13 14 3)(7 8 9 10)$
  - $(3 5)(12 1)(2 6 4)(13 14 3)(7 8 9 10)$
  - So is the answer $\frac{14!}{2!2!}$?

- Another identical permutation:
  - $(1 12)(3 5)(2 6 4)(13 14 3)(7 8 9 10)$
  - So is the answer $\frac{14!}{2!2!2\cdot3\cdot3\cdot4}$?
  - Yes!
Counting Instances of a Type

• In general, the number of permutations of $S_n$ of type $[1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4} ...]$ is

\[
\frac{n!}{\alpha_1! \alpha_2! \alpha_3! \alpha_4! \cdots 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4} \cdots}
\]

Types of $S_5$

<table>
<thead>
<tr>
<th>Type</th>
<th>Example</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1$^5$]</td>
<td>id</td>
<td>1</td>
</tr>
<tr>
<td>[1$^3$] 2$^1$</td>
<td>(1 2)(3)(4)(5)</td>
<td>10</td>
</tr>
<tr>
<td>[1$^2$] 3$^1$</td>
<td>(1 2 3)(4)(5)</td>
<td>20</td>
</tr>
<tr>
<td>[1$^2$] 2$^2$</td>
<td>(1 2)(3 4)(5)</td>
<td>15</td>
</tr>
<tr>
<td>[14]</td>
<td>(1 2 3 4)(5)</td>
<td>30</td>
</tr>
<tr>
<td>[23]</td>
<td>(1 2 3)(4 5)</td>
<td>20</td>
</tr>
<tr>
<td>[5]</td>
<td>(1 2 3 4 5)</td>
<td>24</td>
</tr>
</tbody>
</table>
Conjugate Permutations

- Permutations $\alpha, \beta \in S_n$ are said to be conjugate if there exists $\sigma \in S_n$ such that $\sigma \alpha \sigma^{-1} = \beta$.

- Let $\alpha = (1 \ 2)(3)$ and $\beta = (1)(3 \ 2)$. The two permutations are conjugate, since we can take $\sigma = (1 \ 2 \ 3)$ and $\sigma^{-1} = (3 \ 2 \ 1)$.

Conjugation and Types

- **Theorem.** Two permutations of $S_n$ are conjugate iff they are of the same type.

$$\alpha = (1 \ 2)(3), \ \beta = (1)(3 \ 2), \ \sigma = (1 \ 2 \ 3).$$

$$\sigma \alpha \sigma^{-1} = \beta$$
Proof: One Direction

- Suppose that $\alpha, \beta$ are conjugate, so that $\sigma \alpha \sigma^{-1} = \beta$.
- Consider a cycle $\alpha(x_1) = x_2, \alpha(x_2) = x_3, ..., \alpha(x_k) = x_1$.
- Set $y_i = \sigma(x_i)$. Then $\beta(y_i) = \sigma \alpha \sigma^{-1}(\sigma(x_i)) = \sigma(x_{i+1}) = y_{i+1}$.

Proof: One Direction (cont.)

- $\sigma$ is a bijection between cycles of $\alpha$ and cycles of $\beta$.
- That is, $\alpha$ and $\beta$ are of the same type.
Proof: The Other Direction

- **Suppose $\alpha$ and $\beta$ have the same type.**
  - To prove conjugation, we need to find $\sigma$.
  - Set up a bijection between the cycles of $\alpha$ and $\beta$, so that corresponding cycles have the same length.
  - For every two such cycles $(x_1 \ x_2 \ \ldots \ x_k)$ and $(y_1 \ y_2 \ \ldots \ y_k)$, we set $\sigma(x_i) = y_i$. Then $\sigma \alpha \sigma^{-1}(y_i) = \sigma \alpha(x_i) = \sigma(x_{i+1}) = y_{i+1} = \beta(y_i)$
  - That is, $\sigma \alpha \sigma^{-1} = \beta$.

The End

*So how can we solve this?*

*In the next class, but you can try at home!*