Reminder: Perfect Matchings

• A **perfect matching** of a graph \( G = (V, E) \) is a matching of size \( |V|/2 \).
Reminder: Neighbor Sets

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- For any subset $A \subset V_1$, we define
  
  $N(A) = \{y \in Y \mid (x, y) \in E \text{ for some } x \in A\}$.

\[N(\{b, c, d\}) = \{u, v, w\}\]
\[N(\{a, e\}) = \{u, w, x\}\]

Reminder: Variant of Hall's Theorem

- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- There exists a matching of size $|V_1|$ in $G$ if and only if for every $A \subset V_1$, we have $|A| \leq |N(A)|$.  

Philip Hall
Deficiency

- Let \( G = (V_1 \cup V_2, E) \) be a bipartite graph.
- The deficiency of \( G \) is
  \[
  \text{def}(G) = \max_{A \subset V_1} \{|A| - |N(A)|\}.
  \]
- What is the deficiency of

![Graph with labeled vertices and edges]

\( A = \{b, c, d\} \)

Deficiency

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- The deficiency of \( G \) is
  \[
  \text{def}(G) = \max_{A \subset V_1} \{|A| - |N(A)|\}.
  \]
- The deficiency cannot be smaller than 0 since when \( A = \emptyset \) we have
  \[
  |A| - |N(A)| = 0.
  \]
Deficiency and Maximum Matchings

- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. The size of the maximum matching in $G$ is

\[ |V_1| - \text{def}(G). \]

- This implies **Hall's theorem.**
  - $\text{def}(G) = 0$ if and only if there exists a matching of size $|V_1|$.
  - When $|V_1| = |V_2|$, we have $\text{def}(G) = 0$ if and only if there exists a perfect matching.

**Proof: One Direction**

- Set $d = \text{def}(G)$.
- There exists a subset $A \subset V_1$ such that $|A| - |N(A)| = d$.
- In any matching of $G$, at least $d$ vertices of $A$ are unmatched.
- No matching can have size larger than $|V_1| - d$. 

\[ \begin{array}{c}
  a \\
  b \\
  c \\
  d \\
\end{array} \begin{array}{c}
  w \\
  v \\
\end{array} \]
Proof: One Direction

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- There exists a subset $A \subset V_1$ such that $|A| - |N(A)| = d$.
- In any matching of $G$, at least $d$ vertices of $A$ are unmatched.
- No matching can have size larger than $|V_1| - d$.
- It remains to prove that a matching of this size does exist.

Proof: The Other Direction

- We add $d$ new vertices to $V_2$.
  - We connect every new vertex to each vertex of $V_1$.
- Originally, every set $A \subset V_1$ satisfied $|A| \geq |N(A)| - d$.
  - Now $|A| \geq |N(A)|$. 

```
\begin{array}{cccc}
  a & b & c & v \\
  d & w \\
\end{array}
```

Proof: The Other Direction

- We add $d$ new vertices to $V_2$.
  - We connect every new vertex to each vertex of $V_1$.
- Originally, every set $A \subset V_1$ satisfied $|A| \geq |N(A)| - d$.
  - Now $|A| \geq |N(A)|$.
- By the variant of Hall’s theorem, there exists a matching $M$ of size $|V_1|$.
- Removing the new vertices, we obtain a matching of $G$ of size $|V_1| - d$.

The Size of a Maximum Matching

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- Can we use the deficiency theorem to find the size of the maximum matching of $M$?

- We can check the deficiency of every subset $A \subset V_1$.
  - But there are $2^{|V_1|}$ such subsets!
Alternating Paths

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a matching of $G$.
- A path is alternating for $M$ if every other edge of it is in $M$, and its two extreme vertices are not matched.

\[ p_0 \rightarrow q_1 \rightarrow p_5 \rightarrow q_7 \rightarrow p_{12} \rightarrow q_{12} \rightarrow p_{15} \rightarrow q_{17} \]

Alternating Paths

- A maximal path is alternating for $M$ if every other edge of it is in $M$, and its two extreme vertices are not matched.
- By switching the edges that are in $M$ with the edges that are not, we obtain a larger matching.

\[ p_0 \rightarrow q_1 \rightarrow p_5 \rightarrow q_7 \rightarrow p_{12} \rightarrow q_{12} \rightarrow p_{15} \rightarrow q_{17} \]
Existence of Alternating Paths

Theorem. If a matching $M$ in a bipartite graph $G = (V_1 \cup V_2, E)$ is not a maximum matching, then there exists an alternating path for $M$.

Proof.
- Let $M^*$ be a maximum matching of $G$.
- Let $F$ be the set of edges that are either in $M$ or in $M^*$, but not in both.
- In the graph $G' = (V, F)$, every vertex is of degree at most two.

Example: The Graph $G'$

The graph $G' = (V, F)$.
- Every vertex has a degree of at most two.
- The graph is a union of paths and cycles.
Finding an Alternating Path

- By definition, $M^*$ has more edges than $M$.
- In at least one of the paths of $G'$, $M^*$ has more edges than $M$.
- This must be an alternating path for $M$!

Find a Maximum Matching

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- Start with any matching $M$. A single edge is fine.
- Repeatedly find an alternating path for $M$ and use it to obtain a larger matching.
- The process terminates after at most $|V_1|$ steps.
Finding an Alternating Path

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a non-maximum matching.
- We wish to find whether there is an alternating path for $M$ starting at a specific unmatched vertex $p_0 \in V_1$.
  - We run a variant of BFS from $p_0$.

BFS Variant

- The root of the BFS tree is $p_0$.
- At the first level we have vertices that are adjacent to $p_0$ in $G$. 

![BFS Tree Diagram]
**BFS Variant (2)**

- For each vertex of level 1, if it is matched in $M$, we connect it to its match.

![Diagram of BFS Variant (2)](image)

**BFS Variant (3)**

- For each vertex of level 2, we connect it (by edges not in $M$) to any of its neighbors in $G$ that are not in the tree yet.

![Diagram of BFS Variant (3)](image)
BFS Variant (4)

- We repeat this process:
  - Vertices of even levels ($p_i$'s) have as their children every new vertex adjacent to them.
  - Vertices of odd levels ($q_i$'s) have only their matching vertex as a child.

BFS Variant (5)

- How can we tell whether an alternating path for $M$ starts at $p_0$?
  - Every such path corresponds to an unmatched vertex at an odd level of the tree (i.e., a leaf at an odd level).
Concluding Remarks

- Given a matching $M$ in a bipartite graph $G = (V_1 \cup V_2, E)$, for every vertex of $V_1$ that is unmatched in $M$:
  - Run the BFS variant to check whether there is an alternating path starting from it.
- **If no alternating paths were found** – $M$ is a maximum matching.
- **Otherwise**, we use the alternating path to obtain a larger matching.

A Committee of Committees

- The US senate has 20 committees and each senator may serve on several committees.
- The **committee of committees** should have a representative from each committee, and no senator is allowed to represent more than one committee.
- Is this always possible?
  - No! What if senator Bob is the only person on two committees?
A Committee of Committees?

- How can we find out whether a committee of committees is possible?
  - Build a graph!

A committee of committees is possible if the graph has a matching of size 20.
Problem: Retreat Resort

- **Problem.** A retreat resort currently has \( n \) guests staying in it. On Saturdays, the resort offers hikes with travelling guides.
  - Every guest has a list of hikes that he is interested in.
  - Every guide is allowed to take up to 5 people with him.
  - Describe an efficient algorithm that finds whether every guest can go on a hike that he is interested in.

Building a Graph

- Create a bipartite graph with a vertex for every guest and for every hike.
  - An edge between every guest and every hike that he is interested in.
Fixing the Graph

- A matching in the graph does not take into account that up to 5 people can go on a hike.
- **Split every hike vertex** $v$ **into five vertices**, and connect each of them to each of the vertices that $v$ was connected to.
- There is a valid hiking assignment if and only if the graph has a matching of size $n$.

The End