Ma 2: Homework N.4

due Monday, October 27, 12 noon

1. Consider the Euler equation

\[ t^2 x'' + \alpha t x' + \beta x = 0 \]

for the function \( x = x(t) \) with \( t > 0 \), and with \( \alpha \) and \( \beta \) two real parameters.

- Show that the change of variables \( t = e^u \) transforms the Euler equation into a second order linear equation with constant coefficients for the function \( x = x(u) \).
- Describe the behavior of the solutions of the equation obtained in this way, depending on the real parameters \( \alpha \) and \( \beta \).

2. Solve the following differential equations:

- \( y'' - 2y' + y = e^t/(1 + t^2) \)
- \( y'' - y' - 2y = 2e^{-t} \)

3. Let \( y_1(t) \) and \( y_2(t) \) be two solutions of the homogeneous second order equation

\[ y'' + p(t)y' + q(t)y = 0 \]

where \( p(t) \) and \( q(t) \) are continuous on an interval \( t \in I = (\alpha, \beta) \).

- If the Wronskian of the two solutions is constant, what can one say about \( p(t) \) and \( q(t) \)?
- Show that if \( y_1(t) \) and \( y_2(t) \) vanish at the same point in the interval \( I \), or if they have a maximum or a minimum at the same point, then they are not a fundamental set of solutions.
4. For the following differential equations describe the equilibrium solutions and the asymptotic behavior of the other solutions, for different choices of the initial condition $y(0) = y_0$:

- $\frac{dy}{dt} = e^y - 1$, with initial conditions $-\infty < y_0 < \infty$;

- $\frac{dy}{dt} = y(a - y^2)$, for values of the parameter $a > 0$, $a = 0$, or $a < 0$, and with initial conditions $-\infty < y_0 < \infty$. 