Math 1A Midterm Exam

Due Tuesday, November 4, 12:00 PM noon

Instructions

Please read the following carefully before starting the exam.

- Write your full name on *each* sheet of paper you use for the exam. Enumerate the pages, to make clear which parts belong together.
- Please staple the exam before turning it in and add a cover sheet only carrying your name.
- The exam is to be completed *within 8 hours* of first looking at the questions (not necessarily a consecutive 8 hours, i.e. you can take breaks to eat or such things. Going to a Halloween party does not count as a break), and must be turned in by Tuesday, Nov. 4 at 12:00 PM noon. Use the homework drop box next to the Math Office to turn in the exam once completed.
- As sources you may ONLY use one of the following: Apostol, your lecture notes, and the RESULTS stated as *required* problems in the homework sets. In addition you may also consult your own homework sets 1 - 4, as well as the posted solutions. However, referring to DETAILS of your sets or the solutions posted online is not admissible; if you would like to use, say an idea or a lemma from the latter sources, you have to do write it up again in your exam.
- Generally, you can only use results in Apostol and from the lectures which were covered up to and including on Monday, Oct. 27.
- Please be very explicit in referencing any of the above-mentioned sources.
- No discussion about the exam with anyone (other than me, if something is unclear) is permitted.
- Should something be unclear, please send me an email.

1. Consider a collection of closed intervals $I_n := [a_n, b_n]$, indexed by $n \in \mathbb{N}$, with the property that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, for all $n \in \mathbb{N}$.

(i) Show that the collection satisfies $I_{n+1} \subseteq I_n$ for *all* $n \in \mathbb{N}$, i.e. the intervals I_n are nested.

(ii) Prove that there exists a point x which is in *every* I_n. By giving an example, show that this conclusion is false if we consider *open* intervals instead of closed intervals.

2. Calculate the following limits, or show that they don't exist. Be sure to justify your steps. Values of $\pm\infty$ are allowed. You may use the fact that sin and cos are continuous, although we have not proven this yet.

(i)

$$\lim_{x \to 0} \frac{\tan^2(x)}{x^2}$$

(ii)

$$\lim_{x \to 0^+} \frac{x+2}{\sqrt{x}}$$

(iii)

$$\lim_{x \to +\infty} \frac{\sqrt{x} - \sqrt{x+3}}{5 + \sin(x)}$$

(iv)

$$\lim_{x \to 0} \frac{\sqrt{x^2}}{x}$$

3. Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is continuous and that $f(x) = 0$ for all $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$. Use this to show that if $f, g \colon \mathbb{R} \to \mathbb{R}$ are continuous functions which are equal on all of $\mathbb{Q}$, then in fact they are equal on all of $\mathbb{R}$.

4. Let f be a polynomial of degree n, say $f(x) = \sum_{k=0}^{n} c_k x^k$, such that c_0 and c_n have opposite signs. Prove that $f(x) = 0$ for at least one positive x.

5.

(i) How many *continuous and odd* functions $f : \mathbb{R} \to \mathbb{R}$ can you find which satisfy $(f(x))^2 = x^2$ for all $x \in \mathbb{R}$? You need to, of course, provide an argument justifying your answer.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, continuous function and suppose that $\lim_{x \to \infty} f(x) = 0$. Show that f is bounded and that f has a maximum (on all of $\mathbb{R}$). Give an example showing that such f does *not* necessarily have a minimum (on all of $\mathbb{R}$).