PROBLEM SET NO. 7 (DUE ON MONDAY, NOVEMBER 24 AT 4:00 PM)

- **Problem 1:** (recommended) Let $f : (a,b) \to \mathbb{R}$ be a continuous and strictly monotone function. Show that $f((a,b)) = \{f(x) : x \in (a,b)\}$ is an open interval (possibly unbounded).

  Hint: Distinguish between the cases f is bounded and f is not bounded, and use the intermediate value theorem.

- **Problem 2:** Let I be an open interval. A function $f : I \to \mathbb{R}$ is called **locally invertible at** $x_0 \in I$, if there exists $a < b$ and $c < d$ with $x_0 \in (a,b) \subseteq I$, such that the function

$$g : (a,b) \to (c,d) \ , \ g(x) := f(x) \ , \text{ for } x \in (a,b),$$

  is bijective, in which case g^{-1} is called a local inverse of f near x_0.

  Follow the below-mentioned steps to prove the "**inverse function theorem**" of single variable calculus:

  Given a continuously differentiable function $f : I \to \mathbb{R}$ and $x_0 \in I$. If $f'(x_0) \neq 0$, then f is locally invertible at x_0. Moreover, there exists a local inverse $g^{-1} : (c,d) \to (a,b)$ of f near x_0, such that g and g^{-1} are differentiable and

$$(g^{-1})'(g(x)) = \frac{1}{g'(x)} \ , \ x \in (a,b).$$

  (i) Without loss of generality, assume $f'(x_0) > 0$. Show that $f'(x_0) > 0$ implies that there exists $a < b$ with $x_0 \in (a,b) \subseteq I$ such that $f'(x) > 0$ for all $x \in (a,b)$. A function is continuously differentiable if its derivative is continuous. You'll need this! I've mentioned this definition in class but I'm not sure it's made it onto the board.

  (ii) Taking $c = f(a)$ and $d = f(b)$, combine part (i) and Problem 1 to define g and thus g^{-1}, as claimed in the theorem.

  (iii) Finally, apply the theorem on differentiability of inverse functions proven in class (Apostol, theorem 6.7) to g.

- **Problem 3:** Consider the functions f and g from set 6, problem 3.

  (ii) Show that f is *locally* invertible about $x_0 = 3$ and g is locally invertible about $x_0 = -1$. Argue that the thusly defined *local* inverses, f^{-1} and g^{-1}, are differentiable and find $(f^{-1})'(0)$ and $(g^{-1})'(0)$.

- **Problem 4:** Apostol, Sec. 4.12 problem 30.

- **Problem 5:** The collection of all points (x, y) such that $3x^3 + 4x^2y - xy^2 + 2y^3 = 4$ forms a certain curve $\mathcal{C}$ in the plane. *Assuming* that y can be expressed in the form $y = f(x)$ for some function f, use implicit differentiation to find the equation of the tangent line to this curve $\mathcal{C}$ at the point $(-1, 1)$.
- **Problem 6:** Using the definition of the Riemann-integral, show that if $f \in \mathcal{R}([a, b])$ then, for every $c \in \mathbb{R}$, $c \cdot f \in \mathcal{R}([a, b])$ and

$$\int_a^b (c \cdot f)(x)\mathrm{d}x = c \cdot \int_a^b f(x)\mathrm{d}x \ .$$