Problem 1: (recommended) Let \( f : (a, b) \to \mathbb{R} \) be a continuous and strictly monotone function. Show that \( f((a, b)) = \{f(x) : x \in (a, b)\} \) is an open interval (possibly unbounded).

*Hint: Distinguish between the cases \( f \) is bounded and \( f \) is not bounded, and use the intermediate value theorem.*

Problem 2: Let \( I \) be an open interval. A function \( f : I \to \mathbb{R} \) is called locally invertible at \( x_0 \in I \), if there exists \( a < b \) and \( c < d \) with \( x_0 \in (a, b) \subseteq I \), such that the function

\[
g : (a, b) \to (c, d) \text{, } g(x) := f(x) \text{, for } x \in (a, b),
\]

is bijective, in which case \( g^{-1} \) is called a local inverse of \( f \) near \( x_0 \).

Follow the below-mentioned steps to prove the “inverse function theorem” of single variable calculus:

*Given a continuously differentiable function \( f : I \to \mathbb{R} \) and \( x_0 \in I \). If \( f'(x_0) \neq 0 \), then \( f \) is locally invertible at \( x_0 \). Moreover, there exists a local inverse \( g^{-1} : (c, d) \to (a, b) \) of \( f \) near \( x_0 \), such that \( g \) and \( g^{-1} \) are differentiable and*

\[
(g^{-1})'(g(x)) = \frac{1}{g'(x)} \text{, } x \in (a, b).
\]

(i) Without loss of generality, assume \( f'(x_0) > 0 \). Show that \( f'(x_0) > 0 \) implies that there exists \( a < b \) with \( x_0 \in (a, b) \subseteq I \) such that \( f'(x) > 0 \) for all \( x \in (a, b) \). A function is continuously differentiable if its derivative is continuous. You’ll need this! I’ve mentioned this definition in class but I’m not sure it’s made it onto the board.

(ii) Taking \( c = f(a) \) and \( d = f(b) \), combine part (i) and Problem 1 to define \( g \) and thus \( g^{-1} \), as claimed in the theorem.

(iii) Finally, apply the theorem on differentiability of inverse functions proven in class (Apostol, theorem 6.7) to \( g \).

Problem 3: Consider the functions \( f \) and \( g \) from set 6, problem 3.

(ii) Show that \( f \) is locally invertible about \( x_0 = 3 \) and \( g \) is locally invertible about \( x_0 = -1 \). Argue that the thusly defined local inverses, \( f^{-1} \) and \( g^{-1} \), are differentiable and find \( (f^{-1})'(0) \) and \( (g^{-1})'(0) \).

Problem 4: Apostol, Sec. 4.12 problem 30.
Problem 5: The collection of all points \((x, y)\) such that \(3x^3 + 4x^2y - xy^2 + 2y^3 = 4\) forms a certain curve \(C\) in the plane. Assuming that \(y\) can be expressed in the form \(y = f(x)\) for some function \(f\), use implicit differentiation to find the equation of the tangent line to this curve \(C\) at the point \((-1, 1)\).

Problem 6: Using the definition of the Riemann-integral, show that if \(f \in \mathcal{R}([a, b])\) then, for every \(c \in \mathbb{R}, c \cdot f \in \mathcal{R}([a, b])\) and
\[
\int_a^b (c \cdot f)(x)dx = c \cdot \int_a^b f(x)dx.
\]