Problem 5: Find suitable codomains such that the following functions $f$ defined on $\mathbb{R}$ are surjective. Argue that these functions are in fact invertible by explicitly giving $f^{-1}$.

(i) $f(x) = x^3 + 1$

(ii) $f(x) = \begin{cases} x & \text{if } x \text{ rational,} \\ -x & \text{if } x \text{ irrational.} \end{cases}$

(iii) Given $n$ distinct real numbers $a_1, \ldots, a_n$, $n \in \mathbb{N}$, let

$$f(x) = \begin{cases} x & \text{if } x \neq a_1, \ldots, a_n \\ a_{i+1} & \text{if } x = a_i \text{ for } i = 1, \ldots, n-1 \\ a_1 & \text{if } x = a_n. \end{cases}$$

Hint: For parts (ii) and (iii) based on the definition of $f$ make a guess what $f^{-1}$ could be. Then, verify that this guess is indeed the inverse function of $f$, using the following fact: Given a function $f : A \to B$, where $A, B$ are sets. If there exists a function $g : B \to A$ such that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$, then $f$ is bijective and $g = f^{-1}$. The proof of this fact is short, and just relies on the definition of the inverse; give it a try.

Problem 6: Apostol, Sec. 4.15 problem 8.