PROBLEM SET NO. 3 (DUE ON MONDAY, OCTOBER 20 AT 4:00 PM)

MONDAY, 10-13

Problem 2 may be easier after Tuesday's lecture, but you should think at least think about the different parts a bit now.

- **Problem 1:** (recommended) Consider a collection of closed intervals $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, with the property that $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$, for all $n \in \mathbb{N}$.
  (i) Show that the collection satisfies $I_{n+1} \subseteq I_n$ for *all* $n \in \mathbb{N}$, i.e. the intervals I_n are nested.
  (ii) Prove that there exists a point x which is in *every* I_n. Show that this conclusion is false if we consider open intervals instead of closed intervals.

  The result of part (ii) is called the "*Nested Interval Theorem*". This can be used to show that $\mathbb{R}$ is bigger than $\mathbb{Q}$ even though both are infinite, i.e., there are different sizes of infinity! (One can also show that $\mathbb{N}$ is the same size as $\mathbb{Q}$, even though one is a subset of the other, so the fact about $\mathbb{Q}$ and $\mathbb{R}$ is not obvious!)

  Hint: What can you say about $\alpha := \sup\{a_n : n \in \mathbb{N}\}$ and $\beta := \inf\{b_n : n \in \mathbb{N}\}$? Why do α, β even exist?

- **Problem 2:** Find the value of the following limits (i.e. make an "educated guess"). In each case, use the **ϵ-δ definition** to prove that your proposed value is indeed the limit.
  (i) $\lim_{x\to 1} \dfrac{x^2-1}{x+1}$
  (ii) $\lim_{x\to y} \dfrac{x^n-y^n}{x-y}$, where $n \in \mathbb{N}$ is fixed.
  (iii) $\lim_{h\to 0} \dfrac{\sqrt{a+h}-\sqrt{a}}{h}$, where $a > 0$ is fixed.
  (iv) $\lim_{x\to 0} \dfrac{\sqrt{x}\cos(x)}{1-\sin^2(x)}$

  Hint: For part (iv), use that $|\cos(x)| \geqslant \frac{1}{\sqrt{2}}$ if $|x| \leqslant \frac{\pi}{4}$.
  For part (ii), try to derive the following useful formula:

$$x^n - y^n = (x-y) \cdot \sum_{k=0}^{n-1} x^{n-1-k} y^k \ .$$