PROBLEM SET NO. 2 (DUE ON MONDAY, OCTOBER 14 AT 4:00 PM)

Problems are required unlessed marked as recommended. For the recommended problems, I suggest to just think about a strategy, possibly jotting down some rough ideas, but only work out the details if you have extra time at your disposal.

- **Problem 1:** Let S be a set and A, B, C subsets of S. Prove the following properties of set operations (a-c recommended, d REQUIRED):
  (a) $A \cup B = B \cup A$, $A \cap B = B \cap A$ (*commutative laws*)
  (b) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ (*associative laws*)
  (c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (*distributive laws*)
  (d) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$, $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ (*de Morgan laws*)

  Hint: A useful strategy to show that two sets E, F are equal is to show that both $E \subseteq F$ and $F \subseteq E$ (Why is this true?).
- **Problem 2:**
  (i) Given a set $A \subseteq \mathbb{R}$, in lecture we defined the terms "A is bounded above", "upper bound of A", and "least upper bound (supremum) of A". Using those definitions as a role model, formulate analogous definitions for the terms "*A is bounded below*", "*lower bound of A*", and "greatest lower bound (*infimum*) of A (notation: $\inf A$)". Prove that if $\inf A$ exists, it is unique.
  (ii) Suppose $A \neq \varnothing$ is bounded below. Let B be the set of all lower bounds of A. Show that $B \neq \varnothing$, that B is bounded above, and that $\sup B$ is the greatest lower bound of A.

  Notice that part (ii) in particular shows that the supremum principle guarantees that *every non-empty set $A \subseteq \mathbb{R}$ which is bounded below always has an infimum.*

  Remark: An alternative proof of this fact can be found in Apostol, Theorem I.27.
- **Problem 3:** Let $A \subseteq \mathbb{R}$ such that $\sup A$ exists. Show that an upper bound y of A is the supremum of A *if and only if* for every $\epsilon > 0$, there exists $a \in A$ such that $y - \epsilon < a$.

  Remark: This statement is very useful to determine the supremum of a set, once one is known to exist.