PROBLEM SET NO. 8 (DUE ON FRIDAY, DECEMBER 5 AT 1:00 PM) (NOTE THE DIFFERENT TIME)

- **Problem 1:**
  (a) Let h be continuous and f and g differentiable such that, appealing to the fundamental theorem of calculus (FTC), the function

$$F(x) := \int_{f(x)}^{g(x)} h(t)\mathrm{d}t \ , \tag{1}$$

is well-defined. Show that

$$F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x) \ . \tag{2}$$

  (b) Find the derivatives of each of the following functions:

$$F(x) \ := \ \int_3^{\left(\int_1^x \sin^3 t \ \mathrm{d}t\right)} \frac{1}{1 + \sin^6 t + t^2} \ \mathrm{d}t \tag{3}$$

$$G(x) \ := \ \int_{15}^{x} \left(\int_8^y \frac{1}{1 + t^2 + \sin^2 t} \ \mathrm{d}t \right) \mathrm{d}y \tag{4}$$

Why is the function G at all well-defined (use the FTC)?

- **Problem 2:** Suppose $f \in \mathcal{R}([a, r])$, for all $r > a$. The limit $\lim_{r \to +\infty} \int_a^r f(x)\mathrm{d}x$, if it exists (as a proper limit in $\mathbb{R}$), is denoted by $\int_a^{+\infty} f(x)\mathrm{d}x$, and called an "**improper integral**."
  (a) Determine $\int_1^{+\infty} x^\alpha \ \mathrm{d}x$, if $\alpha < -1$.
  (b) Show that $\int_1^{+\infty} 1/x \ \mathrm{d}x$ does not exist.
  (c) Suppose that $f(x) \geqslant 0$, for $x \geqslant 0$, and that $\int_0^\infty f(x) \ \mathrm{d}x$ exists. Prove that if $0 \leqslant g(x) \leqslant f(x)$, for all $x \geqslant 0$, and if $g \in \mathcal{R}([0, n])$ for each $n \in \mathbb{N}$, then $\int_0^{+\infty} g(x)\mathrm{d}x$ also exists.
  (d) Explain why $\int_0^{+\infty} 1/(1 + x^2) \ \mathrm{d}x$ exists.
  Hint: Split this integral up at 1.

  Hint: For part (c), show first that $\sup_{r \geqslant 0} \int_0^r g(x)\mathrm{d}x$ *exists with* $\sup_{r \geqslant 0} \int_0^r g(x)\mathrm{d}x \leqslant \int_0^\infty f(x)\mathrm{d}x$. *Then, prove that* $\sup_{r \geqslant 0} \int_0^r g(x)\mathrm{d}x = \lim_{r \to +\infty} \int_0^r g(x)\mathrm{d}x$.

- **Problem 3:**
  (a) Suppose f and g are two functions. Give a definition of the function $h(x) := f(x)^{g(x)}$ whenever this makes sense. If f, g are differentiable, compute the derivative of h wherever it exists.

(b) Use part (a) to differentiate the following expressions w.r.t x:
   * 3^x
   * $(\log x)^{\log x}$

- **Problem 4:**
  (i) Prove that for all $n \in \mathbb{N}$,

(5)
$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \leqslant \mathrm{e}^x \text{ , for } x \geqslant 0.$$

  Hint: Use induction on n, and compare derivatives.

  (ii) Use part (i) to conclude that

(6)
$$\lim_{x \to +\infty} \frac{\mathrm{e}^x}{x^n} = +\infty \ .$$

  The limit in (6) shows that the **exponential function grows faster than any polynomial**, which is an important property of the function e^x.

- **Problem 5:** Show that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is bijective. Letting arctan denote the inverse function, derive a formula for $\arctan'(x)$ for $x \in \mathbb{R}$.
- **Problem 6:** Suppose that f is a continuous function (defined for all x) and that the values of the following integrals are known:

  $\int_0^1 f(x)\,dx = 5\,;\ \int_{-1}^1 f(x)\,dx = 3\,;\ \int_0^2 f(x)\,dx = 8\,;\ \int_0^4 f(x)\,dx = 11\,.$

  Evaluate these integrals:

  a) $\int_0^2 f(2x)\,dx$ b) $\int_0^\pi \sin x f(\cos x)\,dx$ c) $\int_2^3 x f(8 - x^2)\,dx$.

  Hint Use substitutions, such as $u = \cos x$ in b).