This is just a general outline for topics we covered during the course. Inclusion in this outline does not guarantee that a topic will be on the final exam, nor does exclusion from this outline guarantee that a topic will not be on the final. *Caveat lector.*

- Introduction and basic definitions
  - Two classical problems
    - What is the area of a figure? How about just, what is the area under a curve?
    - What could we even mean by "instantaneous velocity"?
    - Both problems seem to involve "arbitrarily close" approximations.
  - Numbers
    - How do numbers behave? The field axioms
    - How does size/order work for numbers? The order axioms
    - What makes the reals special? Why are they not just the rationals? The supremum principle
      - If A is a set which is bounded above, $\sup(A)$ is its least upper bound. Recall that $y = \sup(A)$ if for all $\varepsilon > 0$, there is some $a \in A$ such that $y - \varepsilon < a$.
  - Proof techniques
    - Direct proof, i.e. "just do it", e.g. just check the appropriate definitions are satisfied.
    - Proof by negation/contradiction: Assume something holds, show that leads to a contradiction, conclude that thing doesn't hold after all.
    - Induction. Useful for proofs of statements involving the natural numbers.
    - Whenever proving things, keep in mind that the order of quantifiers matters! This is especially important when negating certain statements.
  - Basic facts about the real line
    - The Archimedean property: for every $a \in \mathbb{R}$, there is some natural number n such that $a < n$.
    - The rationals are dense in $\mathbb{R}$. So are the irrationals; this was HW.
  - Functions
    - The formal definition of a function
    - The graph of a function
    - Operations on real-valued functions: addition, multiplication, division (in some cases), composition
    - Special functions: constant functions, identity functions, polynomials
    - Here we discussed trigonometric functions without giving a precise mathematical definition. We covered basic properties and identities.

- Limits and continuity
  - $\varepsilon - \delta$ definition of the limit of a function
    - Basic idea: $\lim_{x \to c} f(x) = z$ means that no matter how close to z you specify, the outputs of f are that close to z as long as the inputs are sufficiently close to c (but not equal to c).

    - The triangle inequality is often useful in finding the value of certain limits.
    - We can use the definition to show that some limits don't exist, e.g. $\lim_{x\to 0}\sin(1/x)$.
  - Arithmetic of limits (a.k.a. the algebra of limits)
  - The Squeeze Theorem
  - Extensions of the limit concept
    - One-sided limits
      - Another test for when a limit exists: $\lim_{x\to c} f(x)$ exists if and only if $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-}$ exist and are equal.
    - Limits involving $+\infty$ or $-\infty$ ("Improper limits")
    - We can still use the arithmetic of limits for these *as long as the limits have values in* $\mathbb{R}$. (Remember, $+\infty$ and $-\infty$ are not real numbers.)
  - Continuity
    - Know the definition. The idea is that if f is continuous at x_0, the behavior of f "close to" x_0 is "consistent" with $f(x_0)$. (So many quote marks! That's why there's a precise mathematical definition.)
    - Examples of continuous functions: polynomials, trig functions, square root
    - Properties of functions which are continuous on a closed interval
      - The Intermediate Value Theorem (This one especially we ended up applying quite often, for instance to show some functions were surjective.)
      - The Boundedness Theorem
      - The Extremal Value Theorem
      - Know why we need to assume our function is continuous and defined on a closed interval.
    - Uniform continuity and the "small-span" problem
      - Functions which are continuous on closed intervals are uniformly continuous. (This was crucial to showing that continuous functions are Riemann integrable.)

- Differentiation
  - Definition of the derivative of a function f at a point x_0
  - Interpretation as the slope of a tangent line, or an instantaneous rate of change
  - How to calculate a derivative using the definition (admittedly, we don't do it often)
  - The arithmetic of derivatives
    - The product rule and the quotient rule are different than they are for limits.
  - The derivative of the composition of two functions, i.e. the chain rule
    - We used this to talk about related rates briefly
  - Higher order derivatives
  - Using the derivative to get information about the function
    - The sign of the derivative and (strict) monotonicity
    - Finding extreme values of functions

        - Optimization problems
  - The Mean Value Theorem
    - This is an important one, we use it a lot.
    - For example, it's how we show that two functions with the same derivative can only differ by adding a constant to one.
    - Also curve-sketching
    - Follows from Rolle's Theorem, a special case
    - L'Hospital's rules are proven using MVT! (Well, the modified form we called the Cauchy Mean Value Theorem.)

- Inverse functions
  - Definitions of injective, surjective, bijective
  - Strictly monotone functions are injective.
  - Using inverse functions to define the square root, or more generally $x \mapsto x^{1/n}$.
  - If $f\colon (a, b) \to (c, d)$ is continuous and bijective, then so is f^{-1}.
    - Also it turns out it's strictly monotone! (Uses IVT.)
  - What if f is differentiable?
    - f^{-1} is not differentiable if and only if $f'(x_0) = 0$. One of those directions was nontrivial! We proved a formula for the derivative of f^{-1}.
    - Implicit differentiation

- Integration
  - Partitions, and their use in approximating the area under a curve
    - The upper and lower sum for a partition
    - Refinements of partitions
  - The definition of (Riemann) integrability
    - The first definition was solely in terms of sup and inf of certain upper and lower sums. We showed that $f \in \mathcal{R}([a, b[)$ if and only if for all $\varepsilon > 0$, there is some partition P_ε such that $\bar{S}(f, P_\varepsilon) - \underline{S}(f, P_\varepsilon) < \varepsilon$.
  - Properties of the Riemann integral
    - Monotonicity
    - Linearity
    - If $a, b, c \in \mathbb{R}$ and f is Riemann integrable on the appropriate intervals, then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.
  - The Fundamental Theorem of Calculus
    - Seriously, this is important.
    - Part 1: If f is continuous on $[a, b]$, then it is Riemann integrable on $[a, b]$.
      - This is not traditionally part of the FTC, but we don't care, we're rebels.
    - Part 2: The area function corresponding to f is an antiderivative for f.

    - Part 3: You can use antiderivatives to calculate the area under a curve.
      - Our two initial problems ended up closely related to one another? Well I'll be.
- Indefinite integrals/antiderivatives
- Logarithms and the exponential function
  - log defined using the area under the graph of $1/x$.
  - exp is just the inverse of log.
  - Properties of exp and log.
- Trigonometric functions
  - We were finally able to give precise definitions of the trig functions, since we could calculate the area in a circular wedge finally.
  - First we defined arccos and used inverses to defined cos. Then we could define $\sin, \arcsin, \arctan$, etc.
  - We calculated the derivative of arccos, and arcsin and arctan are similar.
- Integration techniques
  - Substitution/change of variable
    - This is just the chain rule backwards.
  - Integration by parts
    - This is just the product rule backwards.
  - Partial fraction decompositions
    - Used when integrating rational functions.
    - We had to use substitution, integration by parts, and the fact that $\int \frac{1}{x^2+1}\,dx = \arctan(x)$ to be able to handle all of these.