Lecture 9: \( q \)-hypergeometric functions.

Aims:
- Review hypergeometric functions.
- Introduce generalized basic hypergeometric functions.

9.1. Gauss’s hypergeometric function. Inevitably, any discussion of special functions should make a proper mention to hypergeometric functions. Recall from the first lecture that the classic Gauss’s hypergeometric function is defined by

\[
(9.1) \quad x(1-x) \frac{d^2y}{dx^2} + (c - (a + b + 1)x) \frac{dy}{dx} - aby = 0.
\]

I would like to encourage you to look up more on these functions, such as the characterization of its monodromy group, the algebraic solutions (classified in terms of polyhedra) and the specializations of these functions (including harmonic functions, orthogonal polynomials and elliptic functions).

Let us present a much broader definition which encompasses both hypergeometric and confluent hypergeometric functions, their specializations and their generalizations.

Definition 9.1. A function \( F(x) \) is a hypergeometric function if

\[
F(x) = \sum_{n=0}^{\infty} c_n x^n
\]

if the coefficients, \( c_k \), are such that \( c_0 = 1 \) and for \( n \geq 0 \), they satisfy a recurrence of the type

\[
\frac{c_{n+1}}{c_n} = R(n),
\]

where \( R(n) \) is a rational function whose denominators do not vanish at non-negative integers.

Many examples are functions that are already known to you. For example, if we specify that \( R(x) = \frac{1}{n+1} \) then the coefficients are

\[
c_0 = 1, \quad c_1 = 1, \quad c_2 = 1/2, \quad c_3 = 1/6, \quad \ldots , c_n = \frac{1}{n!}
\]

hence,

\[
y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x
\]
is a hypergeometric function. Similarly, \( R(n) = -1/((n+1)(n+2)) \) we get
\[ c_0 = 1, \quad c_1 = -1/2!, \quad c_2 = 1/4!, \quad c_3 = -1/6!, \ldots, c_n = (-1)^n \frac{1}{(2n)!} \]
hence the function
\[ y(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{(2n)!} = \cos(\sqrt{x}) = \cos(z), \]
may be regarded as a hypergeometric function.

Let us consider the classical Gauss’s hypergeometric function
\[ _2F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) = \sum_{n=0}^{\infty} x^n \frac{(a)_n (b)_n}{(c)_n n!} = \sum_{n=0}^{\infty} x^n c_n, \]
where \((a)_n = \left\{ \begin{array}{ll} 1 & \text{if } n = 0 \\ a(a+1) \ldots (a+n-1) & \text{if } n > 0, \end{array} \right. \)
is called the Pochhammer symbol. The coefficients, \( c_n \), satisfy the recurrence relation
\[ \frac{c_{n+1}}{c_n} = R(n) = \frac{(a+n)(b+n)}{(c+n)(n+1)}. \]
One can see this by substituting \( y = c_k x^k \) directly into (9.1). At the coefficient of \( x^k \) we find
\[ (c+k)(k+1)c_{k+1} - (a+k)(b+k)c_k = 0, \]
as required. An alternatively, another representation, which holds for the general hypergeometric function is the representation for the coefficients, \( c_n \), as the product
\[ c_n = R(0)R(1) \ldots R(n-1). \]

The notation for a generalized hypergeometric function is as follows:
\[ _rF_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_r)_k (b_1)_k \ldots (b_s)_k}{k!} x^k. \]
This definition allows us to express many known special functions as hypergeometric functions.

**Example 9.2.** Let us consider the example from before
\[ _0F_0(-, x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \]
which is simple.

**Example 9.3.** Let us consider the case where we have one parameter, which is a negative integer.
\[ _1F_0 \left( \begin{array}{c} -n \\ - \end{array} ; x \right) = \sum_{k=0}^{\infty} \frac{(-n)_k}{k!} (-x)^k \]
now, notice that \((-n)_k = (-1)^k (n)_k \), \((n)_m = 0 \) for \( m > n \) and \((n)_k = n!/((n-k))! \)
hence
\[ _1F_0 \left( \begin{array}{c} -n \\ - \end{array} ; -x \right) = \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n. \]
This example demonstrates that if one of the $a_k$’s is a negative integer, the resulting hypergeometric function is a polynomial.

**Example 9.4.** We saw that $\cos(x)$ was a hypergeometric function, let us look at

$$x_0 F_1 \left( \frac{-3}{2}; -\frac{x^2}{4} \right) = \sum_{k=0}^{\infty} \frac{1}{(3/2)_k k!} \left(-\frac{x^2}{4}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\frac{3}{2}\right)_k} 2^{2k} k!$$

but look at the denominator, when we absorb $2^k$ into $\left(\frac{3}{2}\right)_k$ we get

$$\left(\frac{3}{2}\right)_k 2^{2k} = (3 \times 5 \times \ldots \times 2k + 1)(2 \times 4 \times \ldots 2k) = (2k + 1)!$$

hence, this is

$$x_0 F_1 \left( \frac{-3}{2}; -\frac{x^2}{4} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sin(x).$$

A similar representation for $\cos(x)$ exists.

**9.2. Basic hypergeometric functions.** We now turn to the $q$-analogue of these special functions.

**Definition 9.5.** A function $F(x)$ is a basic hypergeometric function (also known as a $q$-hypergeometric function) if

$$F(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients, $c_k$, are such that $c_0 = 1$ and for $n \geq 1$, the coefficient satisfy a recurrence of the type

$$\frac{c_{n+1}}{c_n} = R(q^n),$$

where $R(t)$ is a rational function whose denominators do not vanish at $q^k, k \geq 0$.

**Example 9.6.** Let us consider analogously simple examples. When we consider a simple rational function with a linear denominator, we may normalize (through transformations) to give the case

$$R(q^n) = \frac{1 - q}{1 - qx},$$

the coefficients read

$$c_0 = 1, \quad c_1 = \frac{1 - q}{(1 - q)} = 1, \quad c_2 = \frac{(1 - q)^2}{(1 - q)(1 - q^2)} = \frac{1}{[2]!}, \quad \ldots, \quad c_n = \frac{1}{[n]!}$$

meaning that

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = e^x_q$$

is a basic hypergeometric function.

Similarly, if we let the denominator and numerator be linear, we find that

$$R(x) = \frac{x(1 - q)}{(1 - qx)}$$
the the coefficients may be seen to follow the pattern
\[ c_0 = 1, \quad c_1 = \frac{1-q}{1-q} = 1, \quad c_2 = \frac{q(1-q)^2}{(1-q)(1-q^2)} = \frac{1}{[2]^q}, \quad \ldots \quad c_n = \frac{q^n(n-1)/2}{[n]^q}. \]

This means that the resulting hypergeometric function is given by
\[ y(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}x^k}{[k]^q} = E_q^x \]
is a hypergeometric function.

Just like Gauss’s hypergeometric function, we have Heine’s basic hypergeometric function was defined in 1846 by the series representation
\[ _2\phi_1 \left( \frac{q^\alpha}{q^\gamma}; q, x \right) = \sum_{k=0}^{\infty} \frac{(1-q^\alpha)(1-q^\beta)}{(1-q^\gamma)^q}(1-q)^n x^k, \]
which, one can show satisfies the \( q \)-difference equation
\[ (1-x)f(x) - (1+q^{-\gamma} - (q^\alpha + q^\beta)x)f(qx) + (q^{-\gamma} - q^\alpha+\beta)x f(q^2x) = 0. \]
This has the special property that
\[ \lim_{q \to 1} _2\phi_1 \left( \frac{q^\alpha}{q^\gamma}; q, x \right) = _2F_1 \left( \frac{\alpha}{\gamma}; x \right) \]
Most current texts, such as those recommended at the beginning of the course, use the notation \( a = q^\alpha, b = q^\beta \) and \( c = q^\gamma \), which means the equivalent expression for these functions is given by
\[ _2\phi_1 \left( a, b; c, q, x \right) = \sum_{k=0}^{\infty} \frac{(1-a)^n(1-b)^n}{(1-c)^n(1-q)^n} x^k. \]
The equivalent \( q \)-difference equation satisfied by this function is
\[ (1-x)f(x) - (1+c - (a+b)x)f(qx) + (c-abx)f(q^2x) = 0. \]
The generalized basic hypergeometric function is the series of the form
\[ _r\phi_s \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; q, x \right) \]
(9.2)
\[ = \sum_{k=0}^{\infty} \frac{(1-a_1)^k_q \ldots (1-a_r)^k_q (1-s-r)^k_q}{(1-b_1)^k_q \ldots (1-b_s)^k_q} \frac{z^k}{(1-q)^k_q} \]
Analogously to the hypergeometric case, this function encompasses many known functions
\[ _1\phi_0 \left( \frac{0}{-}; q, x \right) = E_q^x, \]
\[ _0\phi_0 \left( \frac{-}{-}; q, -x \right) = E_q^x, \]
We leave the trigonometric cases as assignment exercises.

Many of the expressions involving hypergeometric functions, such as transformation formulas, integral representations and special cases, involve the Gamma
function. Similarly, in the $q$-case, our expressions will necessarily involve the $q$-Gamma function, hence, we present it here as

$$\Gamma_q(x) = \frac{(1 - q)^\infty}{(1 - q^x)^\infty} (1 - q)^{1-x}$$

which has the somewhat desirable property that, in the special case in which the argument is a positive integer,

$$\Gamma_q(n + 1) = 1(1 + q)(1 + q + q^2)\cdots(1 + q + \ldots + q^{n-1})$$

$$= \frac{1 - q}{1 - q} \frac{1 - q^2}{1 - q} \cdots \frac{1 - q^n}{1 - q} = [n]!.$$  

This function also has the nice limit to the normal Gamma function

$$\lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x).$$

We shall explore the Gamma function in more detail later.