Lecture 7: $q$-exponential functions.

Aims:
- Consider the infinite products $(1 + x)^\infty_q$.
- Introduce the two analogues of the $q$-exponential function.

7.1. The $q$-exponential functions and $(1 + x)^\infty_q$. We wish to come to a solution to an important $q$-difference equation

\[ D_q y(x) = y(x) \]  

which is a $q$-analogue of the defining equation for the exponential function. This is not the only $q$-analogue, the other is

\[ D_q y(x) = y(qx) \]  

Both equations limit to the defining equation for the exponential function. These two equations, and their solutions, play important roles in the theory of $q$-series.

Our starting point is the definition

\[ (1 + x)^n_q = \begin{cases} 1 & \text{if } n = 0, \\ (1 + x)(1 + qx)(1 + q^2x) \ldots (1 + q^{n-1}x) & \text{if } n \geq 1. \end{cases} \]

What happens if we let $n \to \infty$? This does not make as much sense without a $q$-deformation, but in the $q$-world, when $|q| < 1$, the infinite product

\[ f(x) = (1 + x)^\infty_q = (1 + x)(1 + qx) \ldots \]

converges to some finite limit. This function clearly satisfies the equation

\[ f(x) = (1 + x)f(qx), \]

hence, $-d_q f(x)$ is given by

\[ -d_q f(x) = f(x) - f(qx) = xf(qx). \]

by dividing by $x(1 - q)$ we obtain the $q$-differential

\[ D_q f(x) = \frac{f(x) - f(qx)}{x(1 - q)} = \frac{f(qx)}{(1 - q)}. \]

We are able to use the special version of the chain rule to see that,

\[ D_q f((1 - q)x) = f((1 - q)qx) \]

which means

\[ y(x) = E^x_q = (1 - (q - 1)x)^\infty_q \]

is a solution to (7.2). Similarly, if we consider

\[ f(x) = \frac{1}{(1 + x)^\infty_q} \]
then the iterate, \(f(qx)\), may be expressed in terms of \(f(x)\) as\
\[
f(qx) = \frac{1}{(1 + x)_q^\infty} = \frac{(1 + x)}{(1 + x)(1 + qx)_q^\infty} = (1 + x)f(x).
\]

By subtracting \(f(x)\) from both sides we obtain\n\[d_q f(x) = f(qx) - f(x) = xf(x)\]
hence, by dividing by \(x(q - 1)\), we obtain the \(q\)-differential equation\n\[D_q f(x) = \frac{f(x)}{q - 1}\]
hence, we may exploit the limited version of the chain rule that we have to notice that\n\[D_q f((q - 1)x) = f((q - 1)x).\]
This means that the solution of (7.1) may be written as\n\[g(x) = e_q^x = \frac{1}{(1 + (q - 1)x)_q^\infty}.\]
The more standard representation is in terms of \((1 - q)\) rather than \((q - 1)\), which we list for convenience\n\[(7.3) \quad e_q^x = \frac{1}{(1 - (1 - q)x)_q^\infty},\]
\[(7.4) \quad E_q^x = (1 + (1 - q)x)_q^\infty.\]
Furthermore, this representation highlights an important relation between the two functions;\n\[(7.5) \quad e_q^x E_q^{-x} = e_q^{-x} E_q^x = 1.\]
We now proceed to provide a series representation for these two functions.

**7.2. The \(q\)-exponential(s).** Now that we are dealing with \((x - a)_q^\infty\), we can redefine the \(q\)-Pochhammer function, for the particular case in which \(x = 1\), as follows\n\[(1 + x)_q^n = \left\{ \begin{array}{ll} 1 & \text{if } n = 0, \\ (1 + x)(1 + qx)(1 + q^2x)\ldots(1 + q^{n-1}x) & \text{if } n \geq 1, \\ (1 + x)(1 + qx)\ldots & \text{if } n = \infty. \end{array} \right.\]
The limited version of Gauss’s binomial theorem states\n\[(1 + x)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{j} x^j.\]
It is also not hard to also show that\n\[\frac{1}{(1 - x)_q^n} = \sum_{j=0}^{n} \frac{[n][n + 1]\ldots[n + j - 1]}{[j]!} x^j.\]
What we wish to do is consider these products in the limit as \(n \to \infty\). Let us deconstruct this as follows, if \(|q| < 1\), then we may legitimately define\n\[\lim_{n \to \infty} \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots = \frac{1}{1 - q}.\]
which allows us to consider the limit of the \(q\)-binomial as \(n \to \infty\)

\[
\lim_{n \to \infty} \binom{n}{j} = \lim_{n \to \infty} \frac{(1-q^n)(1-q^{n-1}) \ldots (1-q^{n-j+1})}{(1-q)(1-q^2) \ldots (1-q^j)}
\]

\[
= \lim_{n \to \infty} \frac{1}{(1-q)(1-q^2) \ldots (1-q^j)}
\]

\[
= \frac{1}{(1-q)^j (1-q)(1-q^2) \ldots (1-q^j)}
\]

\[
= \frac{1}{(1-q)^j [j]!}.
\]

If we use Gauss's and Heine's formula

\[
(7.6) \quad (1 + x)^\infty_q = \lim_{n \to \infty} (1 + x)_q^n = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{(1-q)(1-q^2) \ldots (1-q^j)}
\]

we may substitute \(x\) with \(x(1-q)\) in this expression to obtain a series expansion for \(E^x_q\).

\[
(1 + (1-q)x)^\infty_q = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j(1-q)^j}{(1-q)(1-q^2) \ldots (1-q^j)}
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j q^{j(j-1)/2}}{[1][2] \ldots [j]}
\]

\[
E^x_q = \sum_{j=0}^{\infty} \frac{x^j q^{j(j-1)/2}}{[j]!}
\]

This is our series representation for \(E^x_q\), which bears some similarity to the series representation of \(e^x\), with some added powers of \(q\).

We now turn our attention to the other \(q\)-exponential, as we notice the limit as \(n \to \infty\) of \(1/(1-x)_q^n\) gives us

\[
(7.7) \quad \frac{1}{(1-x)_q^\infty} = \lim_{n \to \infty} \frac{1}{(1-x)_q^n} = \sum_{j=0}^{\infty} \frac{x^j}{(1-q)(1-q^2) \ldots (1-q^j)}
\]

This means that our \(e^x_q\) function is given by

\[
\frac{1}{(1-(1-q)x)_q^\infty} = \sum_{j=0}^{\infty} \frac{x^j(1-q)^j}{(1-q)(1-q^2) \ldots (1-q^j)}
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{[1][2] \ldots [j]}
\]

\[
e^x_q = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}
\]

Recall that

\[
D_q x^n = [n] x^{n-1}
\]
which means that
\[ D_q e^x_q = \sum_{j=0}^{\infty} \frac{D_q x^j_q}{[j]!} = \sum_{j=0}^{\infty} \frac{x^{j-1}}{[j-1]!} = \frac{x^j}{[j]!} \]
and
\[ D_q E^x_q = \sum_{j=0}^{\infty} \frac{q^j (j - 1)}{2} \frac{D_q x^j_q}{[j]!} = \sum_{j=0}^{\infty} \frac{q^j (j - 1)}{2} \frac{q^j x^j}{[j]!} \]
which means that this series representation may also be derived directly from the difference equations
\[ D_q e^x_q = e^x_q, \]
\[ D_q E^x_q = E^{qx}_q. \]

What about the classic addition identity? Does
\[ e^x_q + e^y_q = e^{x+y}_q \]
hold for all \( x \) and \( y \)? We find that the right hand side is given by
\[
\left( \sum_{j=0}^{\infty} \frac{x^j}{[j]!} \right) \left( \sum_{k=0}^{\infty} \frac{y^k}{[k]!} \right) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{[j+k]!}{[j]![k]!} \frac{x^j y^k}{[j+k]!},
\]
by a slight reordering of this sum, by letting \( n = j + k \), then letting \( j \) run from 0 to \( n \) we get
\[
\sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} \right) \frac{1}{[n]!}
\]
but we have seen this before, in the non-commutative setting, when \( yx = qxy \), the bracketed term is also an expression for \((x+y)^n\), hence,
\[ e^x_q e^y_q = \sum_{n=0}^{\infty} \frac{(x+y)^n}{[n]!} = e^{x+y}_q, \]
only when \( yx = qxy \). It is a little bit weird, and very cool, that these functions are somewhat more natural in a non-commutative setting.

### 7.3. Solving scalar q-difference equations.

It is also important to note that this function may also be used to solve any scalar q-difference equation of the form
\[ y(qx) = R(x)y(x) \]
where \( R(x) \) is any rational function of \( x \). First of all, we introduce
\[ \theta_q(x) = (1-q)^\infty_q (1+1/x)^\infty_q (1+qx)^\infty_q, \]
which will explore in more depth next lesson. For now, let us just accept the fact that
\[ \theta_q(qx) = \frac{1}{x} \theta_q(x). \]
Let us proceed by induction, starting with
\[ R(x) = 1 \]
which is satisfied by \( y(x) = 1 \). Given a solution of
\[ y(qx) = R(x)y(x), \]
and given an \( a \in \mathbb{C} \) (including the case where \( a = 0 \)), let us construct the solution of the difference equation

\[
\tilde{y}(qx) = (x - a)^k R(x) \tilde{y}(x).
\]

then let

\[
\tilde{y}(x) = \left( \left( 1 - \frac{q}{x} \frac{a}{x} \right)_q \theta_q(x) \right)^{-k} y(x)
\]

then

\[
\tilde{y}(qx) = \left( \left( 1 - \frac{qa}{qx} \right)_q \theta_q(qx) \right)^{-k} y(qx),
\]

we use the identity

\[
(1 - x)_q^\infty = (1 - x)(1 - qx)_q^\infty
\]

to give

\[
\begin{align*}
&= \left( \left( 1 - \frac{qx}{x} \right)_q \theta_q(qx) \right)^{-k} \left( 1 - \frac{x}{a} \right)^k x^k y(qx) \\
&= (x - a)^k R(x) \left( \left( 1 - \frac{qa}{x} \right)_q \theta_q(qx) \right)^{-k} y(x) \\
&= (x - a)^k R(x) \tilde{y}(x)
\end{align*}
\]

Because any rational function may be expressed as products of this form, we have a solution, by construction, in terms of \((1 - x)_q^\infty\) functions for any \( R(x) \in \mathbb{C}(x) \).