Lecture 6: \(q\)-Fibonacci Numbers

Aims:

- Present the \(q\)-Fibonacci numbers.
- Apply the \(q\)-Pascal’s rule.
- Explore extensions of generating functions for \(q\)-difference equations.

6.1. \(q\)-Fibonacci numbers. We work so hard sometimes it’s nice to goof off and do something silly. To this end, let us consider a \(q\)-analogue of the Fibonacci numbers. The Fibonacci numbers are defined by

\[ f_{n+1} = f_n + f_{n-1}, \]

where \(f_0 = 0, f_1 = 1\). This produces the famous sequence,

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \]

which appear in many contexts.

One of the \(q\)-analogues of this is the set of \(q\)-Fibonacci numbers defined by the recurrence relation

\[
F_n = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
F_{n-1} + q^{n-2}F_{n-2} & \text{if } n > 1.
\end{cases}
\]

This produces a sequence of elements in \(\mathbb{Z}[q]\). Let us compute the first few; clearly \(F_2 = [1]\), giving

\[
F_3 = 1 + q = [2], \\
F_4 = 1 + q + q^2 = [3], \\
F_5 = 1 + q + q^2 + q^3(1 + q) = [5],
\]

making the first few

\[ [0], [1], [1], [2], [3], [5], \]

but, before you get too excited, the next \(q\)-Fibonacci number is


In the limit as \(q \to 1\), we shall show that \(F_n \to f_n\), but, as we can plainly see, \(F_n \neq [f_n]\) for \(n \geq 6\). We will present a general formula in terms of \(q\)-binomials and a generating function, which will be some function satisfied by a \(q\)-difference equation.
6.2. Application of $q$-Pascals rule. Let us consider a famous formula for the Fibonacci numbers, given by

$$f_{n+1} = \sum_{i=0}^{n} \binom{n-i}{i}.$$ 

This has the pictorial representation of summing certain diagonal components of Pascal’s triangle. This is sometimes called the formula for the sums of the “shallow” diagonals.

\begin{center}
\begin{tabular}{ccccccccc}
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 \\
1 & 2 & 1 & 3 & 5 & 8 & 13 & 21 & 34 \\
1 & 3 & 3 & 1 & 5 & 13 & 21 & 34 & 55 \\
1 & 4 & 6 & 4 & 1 & 9 & 28 & 41 & 62 \\
1 & 5 & 10 & 10 & 5 & 1 & 15 & 55 & 90 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & 21 & 77 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 28 & 101 \\
\end{tabular}
\end{center}

**Figure 1.** The pictorial representation of the sum of binomial coefficients that gives the Fibonacci numbers.

The way in which we may prove it is using Pascal’s rule

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

when applied to

$$f_{n+1} = \sum_{i=0}^{n} \binom{n-i}{i} = \sum_{i=0}^{n} \binom{n-i-1}{i} + \sum_{i=0}^{n} \binom{n-i-2}{i-1}$$

now, notice that in the first sum, the term $i = n$ is actually 0, hence, it is actually a sum to $n - 1$, hence, is $f_n$. The second sum, contains neither $i = 0$ or $i = n$, hence, may be considered a sum from $i = 1$ to $n - 1$. By changing the independent variable, $i \rightarrow i + 1$, we get

$$f_{n+1} = f_n + \sum_{i=1}^{n-1} \binom{n-i-1}{i-1} = f_n + \sum_{i=0}^{n-2} \binom{n-i-2}{i} = f_n + f_{n-1}.$$ 

Note that in this representation, most of the summation terms are 0.

The $q$-analogue of this formula is the following

$$F_{n+1} = \sum_{i=0}^{n} q^i \left[ \binom{n-i}{i} \right]_q.$$
We know that the $q$-analogue of Pascal’s rule from the last two lectures is the identity

\[
\begin{aligned}
\binom{n}{j} &= \binom{n-1}{j-1} + q^j \binom{n-1}{j}, \\
\binom{n}{j} &= \binom{n-1}{j} + q^{n-j} \binom{n-1}{j-1},
\end{aligned}
\]

hence, we use the second rule

\[
F_{n+1} = \sum_{i=0}^{n} q^{i^2} \binom{n-i}{i} q
\]

\[= \sum_{i=0}^{n} q^{i^2} \left( \binom{n-i-1}{i} q + q^{n-2i} \binom{n-i-1}{i-1} q \right).\]

Once again, the first summation is actually between $i = 0$ and $n - 1$, and sums to $F_n$. The second term is a sum from $i = 1$ to $n - 1$.

\[
F_{n+1} = F_n + \sum_{i=1}^{n-1} q^{i^2} q^{n-2i} \binom{n-i-1}{i-1} q
\]

\[= F_n + \sum_{i=1}^{n-1} q^{2i-2i} q^n \binom{n-i-1}{i-1} q
\]

which we let $i \to i + 1$, and the sum becomes

\[
F_{n+1} = F_n + q^{n-1} \sum_{i=0}^{n-2} q^{i^2} \binom{n-i-1}{i-1} q = F_n + q^{n-1} F_{n-1}.
\]

When we let $n \to n - 1$, this relation is nothing but

\[
F_n = F_{n-1} + q^{n-2} F_{n-2},
\]

as required. Since, from earlier, we know that

\[
\lim_{q \to 1} \binom{n}{k} = \binom{n}{k},
\]

this also confirms that

\[
\lim_{q \to 1} F_n = f_n.
\]

### 6.3. Generating functions.

The fun doesn’t stop there, we may also consider the generating functions. Let us review the theory around the generating function for the Fibonacci numbers. If we specify that $g(x)$ is the generating function for the Fibonacci numbers, i.e.,

\[
g(x) = \sum_{k=0}^{\infty} f_k x^k
\]
then
\[ xg(x) = \sum_{k=0}^{\infty} f_k x^{k+1} = \sum_{k=1}^{\infty} f_{k-1} x^k, \]
\[ x^2 g(x) = \sum_{k=0}^{\infty} f_k x^{k+2} = \sum_{k=2}^{\infty} f_{k-2} x^k, \]
hence,
\[ g(x) - xg(x) - x^2 g(x) = f_0 + f_1 x - x f_0 + x^2 (f_2 - f_1 - f_2) + x^3 (f_3 - f_2 - f_1) + \ldots \]
We now exploit the fact that \( f_k - f_{k-1} - f_{k-2} = 0 \) for \( k \geq 2 \) to reason that this sum is a finite expression, which only involves \( f_0 \) and \( f_1 \). \( f_0 = 0 \) and \( f_1 = 1 \), in fact, the right hand side is just
\[ g(x) - xg(x) - x^2 g(x) = f_0 + f_1 x - x f_0 = x. \]
Dividing by \( g(x) \) gives us a nice succinct expression for \( g(x) \), which is
\[ g(x) = \frac{x}{1 - x - x^2}. \]
This is also useful in providing the known expression for the Fibonacci numbers in terms of \( \sqrt{5} \), as this expression gives
\[ g(x) = \frac{x}{\sqrt{5}} \left( \frac{1}{x - \frac{1 - \sqrt{5}}{2}} - \frac{1}{x - \frac{1 + \sqrt{5}}{2}} \right), \]
which we expand in \( x \) around \( x = 0 \) to give
\[ g(x) = x \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) x^n. \]
Can we do something similar for the \( q \)-analogue? Unfortunately not. We can get pretty close, by specifying that the generating function satisfies a \( q \)-difference equation, but that is all.
So let us start in the usual way, letting
\[ G(x) = \sum_{k=0}^{\infty} F_k x^k, \]
then certainly
\[ xG(x) = \sum_{k=1}^{\infty} F_{k-1} x^k, \] \[ x^2 G(x) = \sum_{k=2}^{\infty} F_{k-2} x^k \]
but, to make the factor of \( q^{n-2} \), we actually need
\[ x^2 G(qx) = \sum_{k=2}^{\infty} q^k F_{k-2} x^k \]
where we see that
\[ G(x) - xG(x) - \frac{x^2}{q^2} G(qx) = F_0 + x F_1 - x F_0 + x^2 \left( F_2 - F_1 - F_0 \right) + x^3 \left( F_3 - F_2 - q F_1 \right) + \ldots \]
giving us the $q$-difference equation satisfied by the generating function is

$$G(x)(1 - x) - \frac{x^2}{q^2} G(qx) = x$$

Unfortunately, the solutions of this equation are not as simple. We cannot solve this equation quite yet.