Lecture 23: Singularity confinement.

Aims:
- Define singularity confinement for a mapping

Let us consider the QRT mapping arising from the integral

\[ I_n = \frac{1}{x_n} + \frac{1}{x_{n-1}} + x_{n-1} + x_n + \frac{1}{x_n x_{n-1}} \]

so we calculate, \( I_{n+1} - I_n \) with

\[ I_{n+1} - I_n = \frac{1}{x_{n+1}} - \frac{1}{x_n} + x_{n+1} - x_n - \frac{1}{x_n} \left( \frac{1}{x_{n+1}} - \frac{1}{x_{n-1}} \right) = 0 \]

which gives us the equation

\[ (x_{n-1} x_{n+1}) (x_n + 1 - x_{n-1} x_n x_{n+1}) = 0 \]

Now notice that \( x_n = 0 \) is a “bad point” (qualified later) because \( x_{n+1} = \infty \). The singularity confinement property says it may be a “bad point”, but it isn’t that bad because we can still define the evolution past \( x_{n+1} \).

Let us demonstrate by letting \( x_{n-1} \) be free and \( x_n = \epsilon \), then

\[ x_{n+1} = \frac{1 + \epsilon}{x_{n-1} \epsilon}, \]

now let us continue this calculation

\[ x_{n+2} = \frac{1 + \epsilon}{x_{n-1} \epsilon} + \frac{1}{1 + \epsilon} = \frac{1 + x_{n-1}}{1 + \epsilon} \sim \frac{1}{\epsilon} + x_{n-1} + O(\epsilon^2), \]

\[ x_{n+3} = \frac{1 + \epsilon}{\epsilon x_{n-1}} \left( \frac{1 + x_{n-1}}{1 + \epsilon} \right) \sim \epsilon x_{n-1} + O(\epsilon^3), \]

Now something magical happens at \( x_{n+4} \), as

\[ x_{n+4} \sim \frac{1 + \epsilon x_{n-1}}{\epsilon x_{n-1} \left( \frac{1}{\epsilon} + x_{n-1} \right)} + O(\epsilon) \sim \frac{1}{x_{n-1}} + O(\epsilon) \]
To understand why this is magical, let us consider the limits as $\epsilon \to 0$. We have

$$\lim_{\epsilon \to 0} (x_{n+1}) = (x_{n-1}, 0, \infty, 0, \frac{1}{x_{n-1}}, \ldots),$$

which shows us that initial conditions are recovered in a limiting sense.

If we look at $x_n = -1 + \epsilon$, we get the following pattern

$$\lim_{\epsilon \to 0} (x_{n+1}) = (x_{n-1}, -1, 0, \infty, 0, -1, x_{n-1}, \ldots),$$

which also possesses the same property, that the initial conditions are recovered in the limit.

To understand this better, we consider the above system as a map $\phi : \mathbb{C}^2 \to \mathbb{C}^2$, as

$$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ \frac{1+y}{xy} \end{pmatrix},$$

which preserves the integral

$$I = x + y + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}.$$

Let us suppose we are in projective space (so the $x$ or $y$ may be $\infty$), then this map is defined, and

$$\phi : \begin{pmatrix} x_{n-1} \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ \infty \end{pmatrix},$$

the problem here is not so much that the mapping is not defined, at this point, the function is not injective, yet the

$$\phi^4 : \begin{pmatrix} x_{n-1} \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ \frac{1}{x_{n-1}} \end{pmatrix},$$

the fourth power of the map IS injective (or at least is equivalent to an injective regular map around the point). This gives us the following definition.

**Definition 23.1.** A singularity of a map is where the map is not injective.

Recall that locally, a map is injective (has an inverse) if the determinant of Jacobian is non-zero (non-singular). In the above example, taking the Jacobian

$$J = \begin{pmatrix} 0 & 1+y \\ -\frac{1+y}{x^2y} & -\frac{1}{xy^2} \end{pmatrix},$$

hence,

$$\det J = \frac{1+y}{x^2y}.$$

The determinant is singular when $x = 0$ or $y = 0$ or $y = -1$.

**Definition 23.2.** A map possesses the singularity confinement, if for every singularity, $(x, y)$, there exists an $n$ such that $(x, y)$ is not (equivalent to) a singularity of $\phi^n$. 

Let us consider another QRT example with the integral

\[ I_n = (x_n + x_{n-1}) - x_{n-1}x_n(x_{n-1} + x_n) \]

which gives us the system

\[ x_{n+1} + x_n + x_{n-1} = \frac{1}{x_n} \]

with initial conditions \( x_n \) near 0, i.e., \( x_n = \epsilon \), and \( x_{n-1} \) is free. So we have

\[ x_{n+1} = -x_{n-1} - \epsilon + \frac{1}{\epsilon} \]

which tends to \( \infty \) when \( \epsilon \to 0 \). Then

\[
\begin{align*}
x_{n+2} &= x_{n-1} - \frac{1}{\epsilon} + \frac{1}{-x_{n-1} - \epsilon + \frac{1}{\epsilon}} \\
&= -\frac{1}{\epsilon} + x_{n-1} + \epsilon + \epsilon^2 x_{n-1} + O(\epsilon^2)
\end{align*}
\]

where we have used

\[
\frac{1}{-x_{n-1} - \epsilon + \frac{1}{\epsilon}} = \epsilon \left( \frac{1}{1 - \epsilon x_{n-1} - \epsilon^2} \right) \\
\sim \epsilon(1 + \epsilon x_{n-1} + \epsilon^2 + (\epsilon x_{n-1} + \epsilon^2)^2 + \ldots).
\]

We only keep track of to relevant orders\(^1\). If you think the model will be integrable, then you need some sort of expansion to the point that the initial conditions reappear. Next we that almost everything from \(-x_{n+1} - x_{n-2}\) cancel out in \( x_{n+3} \), giving

\[
\begin{align*}
x_{n+3} &= -x_{n+1} - x_{n+2} + \frac{1}{x_{n+2}} \\
&= x_{n-1} + \epsilon - \frac{1}{\epsilon} + \frac{1}{-x_{n-1} - \epsilon - \epsilon^2 x_{n-1}} \\
&\sim -\epsilon^2 x_{n-1} + O(\epsilon^3) + \frac{1}{x_{n+2}} \\
&\sim -\epsilon^2 x_{n-1} + O(\epsilon^3) - \epsilon - \epsilon^2 x_{n-1} \\
&\sim -\epsilon - 2\epsilon^2 x_{n-1}
\end{align*}
\]

\(^1\)We can keep track to higher orders with computer algebra if we need.
Lastly,

\[ x_{n+4} = -x_{n+2} - x_{n+3} + \frac{1}{x_{n+3}} \]

\[ = \frac{1}{\epsilon} - x_{n-1} - \epsilon - \epsilon^2 x_{n-1} + \epsilon + 2\epsilon^2 x_{n-1} + \frac{1}{x_{n+3}} + O(\epsilon^3) \]

\[ = \frac{1}{\epsilon} - x_{n-1} + \frac{1}{-\epsilon - 2\epsilon^2 x_{n-1}} + O(\epsilon^2) \]

\[ = \frac{1}{\epsilon} - x_{n-1} - \frac{1}{\epsilon} \left( \frac{1}{1 + 2\epsilon x_{n-1}} \right) \]

\[ \sim \frac{1}{\epsilon} - x_{n-1} - \frac{1}{\epsilon} + 2x_{n-1} + O(\epsilon^2) \]

\[ \sim x_{n-1} + O(\epsilon) \]

This simply means we have the sequence, as \( \epsilon \to 0 \)

\[ \lim_{\epsilon \to 0} (x_{n+k})_{k=-1}^\infty = (x_{n-1}, 0, \infty, \infty, 0, x_{n-1}, \ldots) \]

which tells us that the fourth power of the map is invertible. The problem with the Jacobian argument is that the Jacobian here is of the form

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & \ast \end{pmatrix} \]

hence, the determinant of the Jacobian is 1. This is the problem with dealing with rational maps as opposed to polynomial maps, where this argument is always true (see Hartshorne).