Lecture 2: Taylor’s Formula.

Aims:
- Present a generalized form of Taylor’s Theorem.

2.1. Taylor’s Theorem. Let us consider a function, \( f \), which possesses all its derivatives at a point \( a \). Such a function is called analytic at \( x = a \). We may reconstruct \( f(x) \) from the derivatives

\[
f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}.
\]

This theorem relies on the functions

\[
P_n(x) = \frac{(x-a)^n}{n!}.
\]

These polynomials have the following properties:
- \( \deg P_n = n \).
- \( P_0 = 1 \) and for \( n > 0 \), \( P(a) = 0 \).
- \( \frac{d}{dx} P_n(x) = P_{n-1}(x) \).

First of all, for any polynomial, \( p(x) \in \Pi_N[x] \), Taylor’s theorem allows us to reconstruct \( p(x) \) around \( x = a \) using the \( \{P_n\} \). For a more general analytic function, this formula is valid on some open disk, \( U \ni a \).

2.2. Generalized Taylor’s Theorem. We now turn to a general Taylor’s theorem, which is general enough to encompass \( D_h \) and \( D_q \), and many other operators (that we don’t consider in this course in detail). We will prove the theorem for polynomials, however, we should stress that this is true for more general functions at least for the particular cases of \( D_q \) and \( D_h \).

**Theorem 2.1.** Let \( a \in \mathbb{C} \) and \( D \) be a linear operator, such that \( D|_{\Pi_N[x]} : \Pi_N[x] \to \Pi_{N-1}[x] \), and \( \{P_0(x), P_1(x), \ldots, \} \) be a sequence of polynomials such that

1. The degree of \( P_n \) in \( x \) is \( n \) for all \( n \).
2. \( P_0(x) = 1 \) and \( P_n(a) = 0 \) for all \( n \geq 1 \).
3. \( DP_n(x) = P_{n-1}(x) \) for all \( n \geq 1 \) and \( DP_0 = D1 = 0 \).

For any polynomial \( f(x) \in \Pi_N[x] \) we may express \( f \) as

\[
f(x) = \sum_{n=0}^{N} D^n f(a) P_n(x)
\]
Proof. Recall that \( \dim \Pi_N[x] = N + 1 \), since \( x^k \) for \( k \in \{0, \ldots, N\} \) forms a basis for \( \Pi_N \). Secondly, the set \( B = \{P_0, P_1(x), \ldots, P_N(x)\} \), are linearly independent functions because they are all of different degrees, hence, \( B \) is another basis for \( \Pi_N[x] \). We may then write

\[
f(x) = \sum_{k=0}^{N} c_k P_k(x)
\]

for unique constants, \( c_k \). If we let \( x = a \), the left hand side reads

\[
f(a) = \sum_{k=0}^{N} c_k P_k(a)
\]

but since \( P_k(a) = 0 \) for \( k \neq 0 \), this reads,

\[
c_0 = f(a).
\]

For any \( 0 < n < N \), we apply \( D^n \) (which is \( D \) applied \( n \) times) then we get that \( D^n \) annihilates \( P_0, \ldots, P_{n-1} \), sends \( P_n(x) \) to 1 and sends \( P_{n+1} \ldots P_N \) to \( P_1(x) \ldots P_{N-n}(x) \). This means that at \( x = a \)

\[
(D_nf)(a) = c_n
\]

which proves (2.1).

Given a linear operator, \( D \), there are two things we need to check:

1. Show that there exists a sequence of polynomials.
2. Show that such a sequence of polynomials is unique

We will do (1), and leave the other as an exercise.

We proceed by proving the existence of the set of polynomials by induction. We know that \( P_0 \) exists, because, \( P_0 = 1 \) by definition.

Let us suppose a set of \( P_k \) exists, with the above specified properties, for all \( k \leq n \), then we apply \( D \) to a test function, which is \( \gamma x^{n+1} + \lambda \), where \( \gamma \) and \( \lambda \) are chosen later. Because we know that \( D \) brings a polynomial of degree \( n + 1 \) to one of degree \( n \), and \( \{P_0, \ldots, P_n\} \) form a basis for the set of polynomials of degree less than \( n \), the application of \( D \) to \( x^{n+1} + \lambda/\gamma \) may be written in terms of the lower \( P_k \), which may be written

\[
\gamma D(x^{n+1} + \lambda/\gamma) = \gamma \left( \sum_{k=0}^{n} b_k P_k(x) \right),
\]

for some constants, \( b_k \). This means, in particular, by letting \( \gamma = 1/b_n \)

\[
D(\gamma x^{n+1} + \lambda) = \gamma \left( b_n P_n(x) + \sum_{k=0}^{n-1} b_k P_k(x) \right),
\]

\[
= P_n(x) + \sum_{k=0}^{n-1} \frac{b_k}{b_n} P_k(x).
\]
Because each $P_k = D(P_{k+1})$, we may write this sum as

$$P_n(x) + \sum_{k=0}^{n-1} \frac{b_k}{b_n} D(P_{k+1}(x)),$$

and bringing the sum over to the other side (by the linearity of $D$) gives us

$$\left( \frac{x^{n+1}}{b_n} + \lambda - \sum_{k=1}^{n} \frac{b_{k-1}}{b_n} P_k(x) \right) = P_n(x).$$

Our candidate for $P_{n+1}$ takes the form

$$P_{n+1}(x) = \frac{x^{n+1}}{b_n} + \lambda - \sum_{k=1}^{n} \frac{b_{k-1}}{b_n} P_k(x),$$

by construction. The final property, $P_{n+1}(a) = 0$ may be satisfied by choosing $\lambda$ correctly; when we let $x = a$,

$$P_{n+1}(a) = \frac{a^{n+1}}{b_n} + \lambda = 0,$$

which implies $\lambda = -\frac{a^{n+1}}{b_n}$. This means that there exists a $P_{n+1}$, by construction, with the right properties.

### 2.3. Taylor polynomials for $D_q$ and $D_h$ around 0.

Let us specialize our theorem to the cases in which $D = D_q$ and $a = 0$. Recall from last lecture that

$$D_q x^n = [n] x^{n-1}.$$

where

$$[n] = 1 + q + \ldots + q^{n-1} = \frac{q^n - 1}{q - 1}.$$

If $[n]$ is a $q$-integer, then we may also define the $[q]$-factorial to be

$$[n]! = \begin{cases} 1 & \text{if } n = 0, \\ [n][n-1]\ldots[1] & \text{if } n > 0. \end{cases}$$

Just as in the case $a = 0$ for the differential case, $P_n = x^n/n!$, we wish to show that

$$P_n = \frac{x^n}{[n]!}.$$

The first property is immediately apparent as $P_n(x)$ is of degree $n$. The second property is simple too, as we have that $x = 0$, then $0^n$ is 0 and naturally have $P_0 = 1$, by definition. The next property, that $DP_n = P_{n-1}$ can be shown directly from

$$D_q P_n = D_q \frac{x^n}{[n]!} = \frac{[n] x^{n-1}}{[n]!} = \frac{x^{n-1}}{[n-1]!} = P_{n-1}(x).$$

The uniqueness tells us that no other choice is possible.
The $D_h$ case is a trickier, but not by much. Let us consider the case for the $D_h$-operator, where $P_0 = 1$ and $P_1 = x/h$. Because we may assume $P_2(0) = 0$, $P_2$ must take the form $P_2 = x(ax + b)$. The first property, that $D_h P_2 = P_1$ gives us

$$D_h P_2 = D_h x(ax + b) = D_h ax^2 + bx = \frac{a(x + h)^2 - ax^2 + b(x + h) - bx}{h} = 2ax + ah + b = \frac{x}{h},$$

giving $a = 1/2h$, then $b = -1/2h$, hence,

$$P_2(x) = \frac{x(x - h)}{2h}.$$

Similarly,

$$P_3(x) = \frac{x(x - h)(x - 2h)}{6h}.$$

It should be clear, and it can be proven, that the general form is

$$P_n = \frac{x(x - h)\ldots(x - (n - 1)h)}{n!h}.$$

The more general form for $x = a$ is part of the first homework.

**3. More general operators (If there is time)**

The two operators, $D_h$ and $D_q$, are

Some more general operators. Let us consider a difference operator

$$D_x f(x) = \frac{f(y_+(x)) - f(y_-(x))}{y_+(x) - y_-(x)}.$$

If $f(x) = 1$ or $f(x) = x$, it is clear that

$$D_x 1 = 0, \quad D_x x = 1.$$

Now we require that $x^2$ is sent to a degree one polynomial, hence

$$D_x x^2 = \frac{y_+(x)^2 - y_-(x)^2}{y_+(x) - y_-(x)} = y_+(x) + y_-(x)$$

which is of degree 1, we let this be $ax^b$. Examining the $x^3$ case, we require that

$$D_x x^3 = \frac{y_+(x)^3 - y_-(x)^3}{y_+(x) - y_-(x)} = y_+(x)^2 + y_+(x)y_-(x) + y_-(x)^2,$$

is degree 2 in $x$. If we subtract $(y_+ + y_-)^2$, then $y_+ y_-$ must be of degree less than or equal to 2, hence, we let

$$y_+ y_- = cx^2 + dx + e.$$

This means

$$(y - y_+)(y - y_-) = y^2 + (y_+ + y_-)y + y_+ y_- = y^2 + (ax + b)y + cx^2 + dx + e,$$

although it is generally written in the literature as

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0,$$
where \( A \neq 0 \). The operators generally send a polynomial of degree \( n \) to \( n - 1 \). We may show that the leading term is

\[
D_x x^n = \frac{n(-B)^{n-1}}{A^{n-1}} x^{n-1} + O(x^{n-2})
\]

We can classify these operators in terms of

\[
\Theta = \frac{(B^2 - AC)(D^2 - AF) - (BD - AE)^2}{A}
\]

and the discriminant \( \Delta = B^2 - AC \).

<table>
<thead>
<tr>
<th>( \Theta )</th>
<th>( \Delta )</th>
<th>conic</th>
<th>lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>parallel lines</td>
<td>linear (( D_h ))</td>
</tr>
<tr>
<td>0 ( \neq 0 )</td>
<td>0</td>
<td>intersecting lines</td>
<td>( q )-linear (( D_q ))</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>0</td>
<td>parabola</td>
<td>quadratic</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td>hyperbola</td>
<td>( q )-quadratic</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>ellipse</td>
<td>( q )-quadratic</td>
</tr>
</tbody>
</table>

When considering the higher lattices, it is also useful to consider an associated \( y \)-coordinate.