Lecture 19: More $q$-Ladder operators.

Aims:
- To find the linearly independent solution to the $q$-difference equation in terms of the associated function.

19.1. $q$-difference-difference equations for orthogonal polynomials.
We now have a system of $q$-difference equations, which we will explore later on in the course in relation to integrable systems. For now, we note that the orthogonal polynomials, $p_n$ and $p_{n-1}$, satisfy a column solution to

\[
Y_n(qx) = \frac{1}{W(x)} \begin{pmatrix} \Omega_n(x) & -a_n \Theta_n(x) \\ a_n \Theta_{n-1}(x) & \Omega_{n-1}(x) - (x - b_{n-1}) \Theta_{n-1}(x) \end{pmatrix} Y_n(x) 
\]

(19.1)

\[
= A_n(x) Y_n(x). 
\]

where

\[
(19.3a) \quad \Theta_n(x) = W(x) p_n(qx) \epsilon_n(x) - V(x) p_n(x) \epsilon_n(qx), 
\]

\[
(19.3b) \quad \Omega_n(x) = a_n W(x) p_n(qx) \epsilon_{n-1}(x) - V(x) p_{n-1}(x) \epsilon_n(qx). 
\]

By the three term recurrence relation, the polynomials also form a solution to

\[
Y_{n+1}(x) = \begin{pmatrix} (x - b_n) \\ 1 \end{pmatrix} \frac{a_{n+1}}{a_n} Y_n(x) - \frac{a_n}{a_{n+1}} Y_n(x) = M_n(x) Y_n(x), 
\]

(19.2)

. Just as in the differential case, there are two equivalent equations for $Y_{n+1}(qx)$, which are given by

\[
Y_{n+1}(qx) = M_n(qx) A_n(x) Y_n(x) = A_{n+1}(x) M_n(x) Y_n(x), 
\]

which is equivalent to the compatibility relation

\[
(19.4) \quad M_n(qx) A_n(x) = A_{n+1}(x) M_n(x). 
\]

On the left hand side, we obtain

\[
\begin{pmatrix} \frac{(qx - b_n)}{a_{n+1}} & -a_n \\ \frac{a_n}{a_{n+1}} & 0 \end{pmatrix} \begin{pmatrix} \Omega_n & -a_n \Theta_n \\ a_n \Theta_{n-1} & \Omega_{n-1} - (x - b_{n-1}) \Theta_{n-1} \end{pmatrix} = 0. 
\]

\[
= \begin{pmatrix} \frac{a_n}{a_{n+1}} \\ -a_n \Theta_n \end{pmatrix} \begin{pmatrix} \Omega_{n+1} & -a_{n+1} \Theta_{n+1} \\ -a_n \Theta_n & \Omega_n - (x - b_n) \Theta_n \end{pmatrix} \frac{a_n}{a_{n+1}} \begin{pmatrix} \frac{(x - b_n)}{a_{n+1}} \\ 0 \end{pmatrix} = 0. 
\]
The bottom entries are both 0, since we use the three term recurrence relation to derive the second row. The first row gives two relations,

\begin{align}
(qx - b_n)\Omega_n - (x - b_n)\Omega_{n+1} - a_{n+1}^2 \Theta_{n+1} + a_n^2 \Theta_{n-1} &= 0, \\
\Omega_{n+1} - \Omega_{n-1} + (x - b_{n-1})\Theta_{n-1} - (qx - b_n)\Theta_n &= 0.
\end{align}

The same asymptotic expansion for $p_n$ and $\epsilon_n$ holds, which we reiterate

\begin{align}
p_n(x) &= \gamma_n \left( x^n - x^{n-1} \sum_{i=0}^{n-1} b_i + O(x^{n-2}) \right) \\
\epsilon_n(x) &= \frac{1}{\gamma_n} \left( x^{-n-1} + x^{-n-2} \sum_{i=0}^{n-1} b_i + O(x^{-n-3}) \right).
\end{align}

This allows us to form bounds on the degree of $\Theta_n$ and $\Omega_n$, which we will consider. Starting with $\Theta_n$, we have

\[ \Theta_n = W \left( \frac{p_n}{n} \right) \epsilon_n - V \left( \frac{p_{n-1}}{n-1} \right) \epsilon_{n-1}, \]

which means the degree of $\Theta_n$ is bounded by

\[ \deg \Theta_n \leq \max(\deg W - 1, \deg V - 1, 0). \]

The bound for $\Omega_n$ follows similarly, as the degrees are

\[ \Omega_n(x) = a_n W \left( \frac{p_n}{n} \right) \epsilon_{n-1} - V \left( \frac{p_{n-1}}{n-1} \right) \epsilon_{n-1}, \]

which gives an upper bound of

\[ \deg \Omega_n \leq \max(\deg W, \deg V - 2, 0). \]

Lastly, we note that we have a $q$-differential equation that results from the above; the

\[ D_q Y_n(x) = \frac{Y(qx) - Y(x)}{x(q - 1)} = \frac{1}{x(q - 1)} (A_n(x) - I) Y_n(x) \]

which will be useful later on because $D_q p_n \sim \gamma [n] x^{n-1}$, which will be useful later on.

19.2. $h$-difference equations for orthogonal polynomials. The entire set of $q$-difference equations for $p_n$ and $\epsilon_n$ were derived with little reference to the underlying action on $x$. For example, if one were to replace the action on $x$ with $x \to x + h$, we obtain a set of $h$-difference equations for $p_n$ and $\epsilon_n$.

Let the linear form defining the orthogonal polynomial ensemble be defined by a sum

\[ L(x) = \sum_{x=a}^{b} w(x) f(x) \]

where $w(x)$ satisfies the difference Pearson’s equation

\[ w(x + h) = \frac{W(x)}{V(x)} w(x), \]

then, by a simple extension of our previous work, we have the following corollary.
Remark 19.1. This condition is often normalized so that $h = 1$, for example, the Hanh polynomials are defined by the linear form

$$L(f(x)) = \sum_{x=0}^{N} \left( \frac{\alpha + x}{x} \right) \left( \frac{\beta + N - x}{N - x} \right) f(x),$$

which may be written as a ratio of Gamma functions.

Suppose the weight function, $w$, satisfies the difference equation (19.7), then the orthogonal polynomials, $p_n$, and associated functions, $\epsilon_n/w$, form the solutions to the linear system of $h$-difference equations

$$Y_n(x + h) = \frac{1}{W(x)} \left( \frac{\Omega_n(x)}{a_n\Theta_{n-1}(x)} \right) Y_n(x),$$

where

$$(19.9a) \quad \Theta_n(x) = W(x)p_n(x + h)\epsilon_n(x) - V(x)p_n(x)\epsilon_n(x + h),$$

$$(19.9b) \quad \Omega_n(x) = a_nW(x)p_n(x + h)\epsilon_{n-1}(x) - V(x)p_{n-1}(x)\epsilon_n(x + h).$$

The associated identities follow naturally from the $h$-difference version of (19.3);

$$(x + h - b_n)\Omega_n - (x - b_n)\Omega_{n+1} - a_n^2\Theta_{n+1} + a_n^2\Theta_{n-1} = 0,$$

$$\Omega_{n+1} - \Omega_{n-1} + (x - b_n)\Theta_{n-1} - (x + h - b_n)\Theta_n = 0.$$

The resulting system could be considered a system of difference-difference equations.

19.3. Discrete $q$-Hermite polynomials. Let us consider, in detail, the case of the $q$-Hermite polynomials, defined by the linear form

$$(f, g) = \int_{-1}^{1} (1 - qx)_q^\infty (1 + qx)_q^\infty f(x)g(x)d_qx.$$

The first few monic versions are

$$H_0 = 1,$$

$$H_1 = x,$$

$$H_2 = x^2 - (1 - q),$$

$$H_3 = x^3 - x(1 - q^2),$$

We want to show

$$p_n(qx) - (1 + q)p_n(x) + q^{-n+1}x^2p_n(x) + qp_n(x/q) = 0.$$ 

Let us start with the weight function

$$w(x) = (1 - qx)_q^\infty (1 + qx)_q^\infty,$$

which we iterate to give

$$w(qx)w(x) = (1 - q^2x)_q^\infty (1 + q^2x)_q^\infty,$$

$$= \frac{(1 - qx)(1 + qx)(1 - q^2x)_q^\infty (1 + q^2x)_q^\infty}{(1 - qx)(1 + qx)},$$

$$= \frac{w(x)}{1 - q^2x^2} = \frac{W(x)}{V(x)}w(x),$$

where

$$V(x) = (1 - qx)(1 - qx)(1 - q^2x)_q^\infty (1 + q^2x)_q^\infty,$$

$$W(x) = (1 - qx)(1 + qx)(1 - q^2x)_q^\infty (1 + q^2x)_q^\infty.$$
which means that a valid choice is
\[ W(x) = 1, \]
\[ V(x) = 1 - q^2 x^2. \]

To derive \( \Omega_n \), consider
\[ \Omega_n = a_n W p_n \epsilon_{n-1} - a_n V \epsilon_n p_{n-1} \]
\[ = a_n \gamma_n \left( q^n x^n + O(x^{n-1}) \right) \frac{1}{\gamma_{n-1}} (x^{-n} + O(x^{-n-1})) \]
\[ - a_n (1 - q^2 x^2) \frac{1}{\gamma_n} \left( q^{-n-1} x^{-n-1} + O(x^{-n-1}) \right) \gamma_{n-1} \left( x^{n-1} + O(x^{n-1}) \right) \]
\[ = q^n - q^{1-n} a_n^2. \]

In the same way, we find an expression for \( \Theta_n \), noting that
\[ \Theta_n = W p_n \epsilon_n - V p_n \epsilon_n \]
\[ = q^n x^n x^{-n-1} - (1 - q^2 x^2) (x^n - \sum_{i=0}^{n-1} b_i + O(x^{n-2})) \times \left( q^{-n-1} x^{-n-1} + q^{-n-2} x^{-n-2} \sum_{i=0}^{n-1} O(x^{-n-2}) \right) \]
\[ = q^{1-n} x - q^{1-n} \sum_{i=0}^{n-1} b_i + q^n \sum_{i=0}^{n} b_i. \]

However, a crucial simplification occurs because the orthogonal polynomial ensemble is odd/even, which can be seen from the weight function satisfying \( w(x) = w(-x) \), the \( b_n \) are 0. This gives us
\[ \Theta_n = q^{1-n} x. \]

This gives us that the \( A_n(x) \) is given by
\[ A_n(x) = \left( q^n + q^{1-n} a_n^2 \right) x^n a_n \left( q^{n-1} - q^{2-n} x^2 \right) + q^{2-n} a_n^2 \left( q^{n-1} - q^{2-n} x^2 + q^{2-n} a_n \right). \]

We will continue the discussion of this case in the next lecture.