Lecture 10: Sums and transformation formulas

Aims:
- Exhibit some basic hypergeometric transformations.
- Demonstrate a summation formula.

10.1. Heine’s Transformation formula. A basic hypergeometric representation for a given function is by no means unique. There are groups of transformation between various hypergeometric representations of the same function. We will first prove the classical Heine’s transformation formula which will be useful in proving many other formulas.

**Theorem 10.1 (Heine’s transformation formula).**

\[ _2\phi_1 \left( \begin{array}{c} a, b \\ c \\ \end{array}; q, z \right) = \frac{(1 - b)_q \infty (1 - az)_q \infty}{(1 - c)_q \infty (1 - z)_q \infty} _2\phi_1 \left( \begin{array}{c} c/b, z \\ az \\ \end{array}; q, b \right) \]

**Proof.** The main ingredient for this proof is a question from the homework exercises:

\[ (1 - az)_q \infty (1 - z)_q \infty = \sum_{k=0}^{\infty} \frac{(1 - a)^k}{(1 - q)^k} z^k. \]

Now recall the basic identity

\[ (1 - a)_q \infty = (1 - a)(1 - aq)(1 - aq^{n-1})(1 - aq^n)_q \infty, \]

hence, we may write

\[ _2\phi_1 \left( \begin{array}{c} a, b \\ c \\ \end{array}; q, z \right) = \sum_{n=0}^{\infty} \frac{(1 - a)^n}{(1 - c)^n} \frac{(1 - b)^n}{(1 - q)^n} z^n = \frac{(1 - b)_q \infty}{(1 - c)_q \infty} \sum_{n=0}^{\infty} \frac{(1 - a)^n}{(1 - q)^n} (1 - cq^n)_q \infty z^n. \]

Now using the homework question for \( a = c/b \) and \( x = bq^n \), giving

\[ = \frac{(1 - b)_q \infty}{(1 - c)_q \infty} \sum_{n=0}^{\infty} \frac{(1 - a)^n}{(1 - q)^n} \sum_{m=0}^{\infty} \frac{(1 - c/b)^m}{(1 - q)^m} (bq^n)^m \]

now, swapping the sums around we get

\[ = \frac{(1 - b)_q \infty}{(1 - c)_q \infty} \sum_{m=0}^{\infty} \frac{(1 - c/b)^m}{(1 - q)^m} b^m \sum_{n=0}^{\infty} \frac{(1 - a)^n}{(1 - q)^n} (zq^m)^n \]
but this sum, is just the homework question, with \( x = zq^m \), which gives us
\[
\left( 1 - b \right)_q^\infty \sum_{m=0}^\infty \frac{(1-c/b)_q^m b^m}{(1-zq^m)_q^\infty} (1 - azq^m)_q^\infty
\]
which proves the formula.

\[\square\]

10.2. \( q \)-Vandermonde sums. The usual Vandermonde identity usually states that
\[
\binom{n+m}{k} = \sum_j \binom{n}{j} \binom{m}{k-j}
\]
and it can be seen to arise from Pascal’s rule in the following way; if we consider the left hand side
\[
\binom{n+m}{k} = \binom{n+m-1}{k} + \binom{n+m-1}{k-1}
\]
if we repeat the process, we get
\[
\binom{n+m}{k} = \binom{n+m-2}{k} + 2 \binom{n+m-2}{k-1} + \binom{n+m-2}{k-2}
\]
and so on. The coefficient of the right hand side are, by no coincidence, binomial coefficients themselves. Alternatively, this comes from comparing the coefficients of
\[
(1 + x)^{m+n} = (1 + x)^m(1 + x)^n,
\]
which can be easily seen.

The hypergeometric representation of this identity is
\[
_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},
\]
which specializes to the above when \( a, b \) and \( c \) are integers. The \( q \)-analogue of this is as follows.

**Proposition 10.2** (Heine’s Summation formula).
\[
_2\phi_1 \left( \begin{array}{c} a, b \\ c \end{array} : q, \frac{c}{ab} \right) = \frac{(1-c/b)_q^\infty (1-c/b)_q^\infty}{(1-c)_q^\infty (1-c/ab)_q^\infty}
\]

**Proof.** Letting \( x = c/ab \) in Heine’s transformation formula
\[
_2\phi_1 \left( \begin{array}{c} a, b \\ c \end{array} : q, \frac{c}{ab} \right) = \frac{(1-b)_q^\infty (1-c/b)_q^\infty}{(1-c)_q^\infty (1-c/ab)_q^\infty} _2\phi_1 \left( \begin{array}{c} c/b, c/ab \\ c/b \end{array} : q, b \right)
\]
\[
= \frac{(1-b)_q^\infty (1-c/b)_q^\infty}{(1-c)_q^\infty (1-c/ab)_q^\infty} _1\phi_1 \left( \begin{array}{c} c/ab \\ c/ab \end{array} : q, b \right)
\]
\[
= \frac{(1-b)_q^\infty (1-c/b)_q^\infty}{(1-c)_q^\infty (1-c/ab)_q^\infty} \sum_{n=0}^\infty \frac{(1-c/ab)_q^n b^n}{(1-q)_q^n}
\]
but this sum is yet another application of the homework question, where \( x = b \) and \( a = c/ab \), which gives
\[
2\phi_1 \left( \frac{a, b}{c}; q, \frac{c}{ab} \right) = \frac{(1 - b)_q^\infty (1 - c/b)_q^\infty (1 - c/a)_q^\infty}{(1 - c)_q^\infty (1 - c/ab)_q^\infty (1 - b)_q^\infty}.
\]
which completes the proof.

**Corollary 10.3 (\( q \)-Chu-Vandermode identity).**
\[
2\phi_1 \left( \frac{a, q^{-n}}{c}; q, q \cdot n \right) = \frac{(1 - c/a)_q^n}{(1 - c)_q^n (1 - q^n c/a)_q^\infty}
\]

**Proof.** By letting \( b = q^{-n} \), this formula becomes
\[
2\phi_1 \left( \frac{a, q^{-n}}{c}; q, \frac{q^n c}{a} \right) = \frac{(1 - c/a)_q^n (1 - q^n c)_q^\infty}{(1 - c)_q^n (1 - q^n c/a)_q^\infty}
\]
but recalling
\[
(1 - c)_q^\infty = (1 - c)_q^n (1 - q^n c)_q^\infty
\]
we have
\[
2\phi_1 \left( \frac{a, q^{-n}}{c}; q, \frac{q^n c}{a} \right) = \frac{(1 - c/a)_q^n}{(1 - c)_q^n}.
\]

**10.3. Jackson’s transformations.** The second transformation formula requires use of the \( q \)-Vandermonde identity above. It is also a very useful identity in its own right.

**Proposition 10.4.**
\[
2\phi_1 \left( \frac{a, b}{c}; q, z \right) = \frac{(1 - ax)_q^\infty}{(1 - x)_q^\infty} 2\phi_2 \left( \frac{a, c/b}{c, az}; q, bz \right)
\]

**Proof.** Let us first use the \( q \)-Vandermonde identity above to write
\[
\frac{(1 - b)_q^k}{(1 - c)_q^k} = \sum_{n=0}^{\infty} \frac{(1 - q^{-k})_q^n (1 - c/b)_q^n}{(1 - c)_q^n (1 - q)_q^n} (bq^k)^n
\]
in
\[
2\phi_1 \left( \frac{a, b}{c}; q, z \right) = \sum_{k=0}^{\infty} \frac{(1 - a)_q^k (1 - b)_q^k}{(1 - c)_q^k (1 - q)_q^k} z^k
\]
\[
= \sum_{k=0}^{\infty} \frac{z^k (1 - a)_q^k}{(1 - q)_q^k} \sum_{n=0}^{k} \frac{(1 - q^{-k})_q^n (1 - c/b)_q^n}{(1 - c)_q^n (1 - q)_q^n} (bq^k)^n
\]
now we need to realize that

\[(1 - q^{-k})^n = (1 - q^{-k})(1 - q^{-k+1}) \ldots (1 - q^{-k+n-1})\]

\[= q^{-nk}(q^k - 1)(q^k - q) \ldots (q^k - q^{n-1})\]

\[= q^{-nk} q^{n(n-1)/2} (q^k - 1)(q^{k-1} - 1) \ldots (q^{k-n+1} - 1)\]

\[= (-1)^n q^{-nk} q^{n(n-1)/2} (1 - q^{n-k+1}) \ldots (1 - q^k)\]

\[= (-1)^n q^{-nk} q^{n(n-1)/2} \frac{(1 - q)^k}{(1 - q)^{k-n}}\]

which we substitute into the above and swap the summation (note that now it’s
doubly infinite, but most of the summation terms are 0) to obtain

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1 - a)^n (1 - c/b)^n}{(1 - q)^k q^n (1 - q)^n (1 - c)^n q^n} (-b)^n z^k q^n(n-1)/2\]

now we shift \(k \to k + n\), which has the effect

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1 - a)^n (1 - c/b)^n}{(1 - q)^k q^n (1 - q)^n (1 - c)^n q^n} (-b z)^n q^n(n-1)/2 \sum_{k=0}^{\infty} \frac{(1 - a q^n)^k}{(1 - q)^n} z^k\]

but \((1 - a)^{n+k} = (1 - a)^n (1 - a q^n)^k\), giving

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1 - a)^n (1 - c/b)^n}{(1 - q)^n (1 - c)^n q^n} (-b z)^n q^n(n-1)/2 \sum_{k=0}^{\infty} \frac{(1 - a q^n)^k}{(1 - q)^n} z^k\]

This last sum is yet another application of the homework question, which gives us

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1 - a)^n (1 - c/b)^n}{(1 - q)^n (1 - c)^n q^n} (-b z)^n q^n(n-1)/2 \frac{(1 - a q^n z)^\infty}{(1 - z)^\infty}\]

Lastly, we note, once more

\[(1 - a z)^\infty = (1 - a z)^n (1 - a q^n)^\infty\]

which means we finally have

\[= \frac{(1 - a z)^\infty}{(1 - z)^\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1 - a)^n (1 - c/b)^n}{(1 - c)^n q^n (1 - a z)^n q^n} (-b z)^n q^n(n-1)/2\]

\[= \frac{(1 - a z)^\infty}{(1 - z)^\infty} 2f_2 \left( \frac{a, c/b}{c, a z}; q, b z \right)\]

completing the proof. □

Some of the consequences of this transformation have been left an as exercise.