Homework 3: Orthogonal polynomials, Bessel and Airy functions.

1. The Legendre polynomials are orthogonal with respect to the bilinear form

\[ \langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx \]

(a) Show that \( b_n = 0 \) for all \( n \).

(b) The weight function may be considered to satisfy \( w'/w = 2V/W \) where \( W(x) = 1 - x^2 \) and \( V(x) = 0 \). Using these values show that the Legendre polynomials satisfy the differential equation

\[
(1 - x^2) \frac{d}{dx} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} -nx & a_n(2n+1) \\ -a_n(2n-1) & nx \end{pmatrix} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}
\]

(c) Find \( a_1^2 \) and use the compatibility between the three term recurrence relation and the above differential equation to deduce that

\[ a_n^2 = \frac{n^2}{(2n-1)(2n+1)}. \]

(d) Use the above results to show that the Legendre polynomials solve

\[
\frac{d}{dx} \left((1 - x^2) \frac{d}{dx} p_n(x)\right) + n(n+1)p_n(x) = 0.
\]

(e) Using the differential equation, show that the Legendre polynomials are proportional to

\[ p_n(x) \propto \, _2F_1 \left( -n, n+1 \mid \frac{1-x}{2} \right) \]

(f) Using the ladder operator, show that for all natural numbers, \( n \) and \( m \), such that \( n \neq m \),

\[
\int_0^1 \, _2F_1 \left( -n, n+1 \mid x \right) \, _2F_1 \left( -m, m+1 \mid x \right) dx = 0.
\]

2. The little \( q \)-Legendre are orthogonal with respect to the bilinear form

\[ \langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)d_qx = (1 - q) \sum_{k=0}^{\infty} q^k f(q^k)g(q^k) \]
(a) Show using the initial values of $a_0 = 0$ and $b_0 = 0$ that

\[ b_n = \frac{2q^n[n][n+1]}{[2n][2n+2]}, \]

\[ a_n^2 = \frac{q^{2n-1}[n]^4}{[2n-1][2n]^2[2n+1]}, \]

hence, show that the polynomials satisfy the matrix difference equation

\[
\begin{pmatrix}
    p_n(qx) \\
    p_{n-1}(qx)
\end{pmatrix} = \frac{1}{1 - qx} \left( \begin{array}{c}
    \frac{2q^n}{1 + q^n} - q^{n+1}x \\
    (q^{1-n} - q^n)a_n \\
\end{array} \right) \left( \begin{array}{c}
    a_n(q^{n+1} - q^{-n}) \\
    2 \\
    1 + q^n - q^{1-n}x
\end{array} \right) \left( \begin{array}{c}
    p_n(x) \\
    p_{n-1}(x)
\end{array} \right)
\]

(b) Use part (a) to show that $p_n(x)$ satisfies

\[
D_q \left( x(1-x)D_q \left( p_n \left( \frac{x}{q} \right) \right) \right) + q^{-n}[n][n+1]p_n(x) = 0.
\]

(c) Use the above $q$-differential equation to show that

\[ p_n(x) \propto {}_2\phi_1 \left( \begin{array}{c}
    q^{-n}, q^{n+1} \\
    q
\end{array} ; q, qx \right)
\]

(d) Use the $q$-difference equation, show that

\[
\int_0^1 {}_2\phi_1 \left( \begin{array}{c}
    q^{-n}, q^{n+1} \\
    q
\end{array} ; q, qx \right) 2\phi_1 \left( \begin{array}{c}
    q^{-m}, q^{m+1} \\
    q
\end{array} ; q, qx \right) = 0
\]

when $n \neq m$.

3. We consider the second $q$-Bessel function, which satisfies the equation

\[
(1 + \frac{q^2 x^2}{4}) u(q^2 x) - (q^n + q^{-n}) u(qx) + u(x) = 0
\]

(a) Show that the second $q$-Bessel function admits the basic hypergeometric representation

\[
J_{\nu}^{(2)}(x) = \frac{(x/2)^\nu (1 - q^{2\nu+2})_\infty}{(1 - q^2)_\infty q^2} 2\phi_1 \left( \begin{array}{c}
    - \\
    q^{2\nu+2} : q^2, -\frac{x^2 q^{2\nu+2}}{4}
\end{array} \right).
\]

(You should assume the same prefactor as the first $q$-Bessel function)

(b) Show that the following function,

\[ u(x) = \left( 1 - \frac{x^2}{4} \right)_\infty J_{\nu}^{(1)}(x), \]

written in terms of the first $q$-Bessel function, satisfies the $q$-difference equation for the second $q$-Bessel function above.