MA 110a Fall 2013

PROBLEM SET 8
Due on Friday, December 6

1. Let $X$ be a normed space, and let $\{x_n\} \subset X$, $\{f_n\} \subset X'$ (dual space). If $x_n \to x$ and $f_n \rightharpoonup^w f$, does it follow that $f_n(x_n) \to f(x)$? What if $x_n \rightharpoonup^w x$ and $f_n \to f$?

2. Let $X$ be a normed space. If $X'$ is separable, then $X$ is separable.

3. The space $(l^2)$ with weak topology is first category in itself.

4. Consider weak, weak*, and norm topologies on the unit ball $B \subset l^1$. Are they all different?

5. Let $X$ be a compact Hausdorff space. Consider the unit ball in $C(X)'$. Show that the extremal points are precisely the functionals $f \mapsto \zeta f(x), x \in X, |\zeta| = 1$. (Problem 5.12.4)

6. Let $X$ be a compact Hausdorff space and let $T : C(X) \to C(X)$ be an isomorphism (i.e. $T$ is invertible, preserves norms). Then

$$(Tf)(x) = a(x)f(\phi(x)), \quad (f \in C(X), \ x \in X),$$

for some unimodular continuous function $a : X \to \mathbb{C}$ and some homeomorphism $\phi : X \to X$.

7. Let $H$ be a Hilbert space, let $a \in H$, and let $A$ be a bounded operator in $H$. Show that the function

$$x \mapsto \|Ax - a\|^2, \quad (x \in H),$$

attains its minimal value on every bounded closed convex set in $H$.

8. Problem 5.11.5