Some easy and some hard problems about graphs

Suppose $G$ is a large graph. For example, you may imagine that $G$ has 10000 or so vertices and average degree 100 or 200. Here are four questions I might ask about $G$.

Q0. Are 1000-digit numbers $a$ and $b$ coprime? Justify your answer.
Q1. Is $G$ bipartite? Justify your answer.
Q2. Is $G$ Eulerian? Justify your answer.
Q4. Is $\chi(G) \leq 3$? Justify your answer.

What do I mean when I say “justify”? Well, I expect data that I can check “easily” on my computer in a few seconds or minutes. It is not enough to say that you and your computer stayed up all night and e.g. checked all $2^{10000}$ ways the vertices of $G$ can be colored red and blue.

If your answer to Q0 is NO, you should provide me with a common divisor $d > 1$. I can check that $d$ divides both $a$ and $b$ quickly. If your answer is YES, please supply me with integers $s$ and $t$ so that $sa + tb = 1$. I can check this much faster than I could check the perhaps thousands of steps of the Euclidean algorithm that you had to do.

If your answer to Q1 is YES, I want the list of red vertices in a proper 2-coloring of $G$. Then I can go through the million edges and check that each has one red end and one blue end. If your answer is NO, I want you to list the vertex sequence of a cycle of odd length. I can check that successive vertices are adjacent and all are distinct (save the first is equal to the last).

An Eulerian walk in a graph $G$ is a walk that traverses every edge exactly once. A graph $G$ is Eulerian when there is a closed Eulerian walk in $G$. A theorem we will talk about later states that a connected graph $G$ is Eulerian if and only if all vertices of $G$ have even degree.

If your answer to Q2 is YES, I want the vertex sequence of the closed walk. Then I can go through the million edges and check that each is traversed once. If your answer is NO, I want you to specify a vertex $x$ that has odd degree. I can check that really quickly.

An Hamiltonian cycle in a graph $G$ is a cycle that traverses every vertex. A graph $G$ is Hamiltonian when there is a Hamiltonian cycle in $G$. Examples: The cube graph is Hamiltonian. The Petersen graph is not Hamiltonian.

If your answer to Q3 is YES, I want the vertex sequence of the Hamiltonian cycle. If your answer is NO, no one knows what you can do, in general.

If your answer to Q4 is YES, I want the lists of red, blue, and green vertices in a proper coloring (not all colors need to be used). If your answer is NO, no one knows what you can do, in general. (You could be lucky and notice that $G$ has a complete subgraph on four vertices—that would be definitive.)
The problems Q3 and Q4 are known to be among the hardest computational problems in the world. Search for “NP-complete” on the web. We will talk about algorithms for Q1 ad Q2 later.

The greedy algorithm

The greedy algorithm is named after Professor P. Q. Greedy. This is a joke.

There are a variety of algorithms called greedy algorithms. See e.g. the the Wikipedia article.

We describe the classical greedy coloring algorithm for graphs. As colors, we use the natural numbers 1, 2, 3,... . Given a graph $G$ with $n$ vertices, order (or enumerate) the vertices as $x_1, x_2, \ldots, x_n$. Assign $x_1$ the color 1. After $x_1, x_2, \ldots, x_k$ have been assigned colors, $k < n$, give $x_{k+1}$ the color $j$ that is the smallest color (integer) that does not appear on any of the neighbors of $x_{k+1}$. Stop when $x_n$ has been colored.

It should be clear that every vertex has been colored and that we have a proper coloring. The greedy algorithm does not directly answer a YES-NO question, e.g. whether or not $\chi(G) \leq 3$. If we are lucky, we may have used relatively few colors. (We found in class a proper coloring of the 10 vertices of the Petersen graph with three colors, using one particular ordering of its vertices.) But greedy is not always the best strategy.

**Proposition.** If $G$ is a graph in which all vertices have degrees $\leq k$ for some positive integer $k$, then $\chi(G) \leq k + 1$.

**Proof:** Choose any ordering of the vertices and use the greedy coloring algorithm. $\square$