This section of the notes may be extended later. For the moment, statements and explanations are very brief. See Biggs

**Permutation groups; cycle decomposition**

A *permutation* of a set $S$ is a bijection $\sigma : S \to S$. The *product* $\sigma \tau$ of two permutations $\sigma$ and $\tau$ of $S$ is defined by $(\sigma \tau)(x) = \sigma(\tau(x))$. Remember that $\sigma \tau$ means “use $\tau$ first, then $\sigma$”.

Given a permutation $\pi$ of set $X$, consider the digraph $D$ whose vertex set is $X$ and whose edges are $(x, \pi(x))$, $x \in X$. Because $\pi$ is a mapping, every vertex has outdegree 1. Because $\pi$ is also a permutation, every vertex has indegree 1. As an unirected graph, $D$ is regular of degree 2 and all the components are polygons. In $D$, the components are (digraphs of) directed cycles, and they are, in my mind, what the cycles of $\pi$ really are. But the following language and notation is very common (and useful when we need to write in lines of type rather than diagrams).

A *cycle* on $S$ is a permutation of $S$ that permutes some of the elements cyclically and fixes all others. That is, a permutation of the form $\pi = (a_1 \ a_2 \ a_3 \ \ldots \ a_k)$ where this notation means that

\[
\pi(a_1) = a_2, \quad \pi(a_2) = a_3, \quad \ldots, \quad \pi(a_{k-1}) = a_k, \quad \pi(a_k) = a_1,
\]

and where $\pi(x) = x$ for any $x \not\in \{a_1, a_2, \ldots, a_k\}$. (We are assuming that the $a_i$’s are distinct.)

**Proposition 1.** Every permutation can be written uniquely as the product of disjoint cycles.

**Example.** Let $\sigma$ be the permutation described in “two-row notation” by

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
5 & 6 & 1 & 7 & 9 & 2 & 8 & 3 & 4
\end{pmatrix}.
\]

Then $\sigma = (1 \ 5 \ 9 \ 4 \ 7 \ 8 \ 3)(2 \ 6)$.

The *order* of a permutation $\pi$ is the least positive integer $n$ so that $\pi^n = id$. One can see that the order of a permutation of a finite set is the LCM of the lengths of its cycles. Our example above has order 14.

**Permutation groups and the Orbit-Stabilizer Lemma**

A set $G$ of permutations is a *group* or a group of permutations of a set $X$ when it is nonempty and closed under multiplication of permutations. That is, when $\sigma \tau \in G$ whenever $\sigma \in G$ and $\tau \in G$. 
Examples include
(i) the set $\langle \sigma \rangle$ of all powers $\{id = \sigma^0, \sigma, \sigma^2, \ldots \}$ of a single permutation,
(ii) the set $\text{SYM}(X)$ of all permutations of a set $X$, and
(iii) the set of automorphisms of a graph (to be defined later).

Let $G$ be a group of permutations of a set $X$. For $x \in X$, the orbit of $x$ is the subset $O_x$ of $X$ defined by
$$O_x = \{\sigma(x) : \sigma \in G\}.$$  

The stabilizer of $x$ is the subset $\{\sigma \in G : \sigma(x) = x\}$ of $G$, and is denoted by $G_x$ or by $G(x \mapsto x)$.

Fact: If $y \in O_x$, then $O_y = O_x$; equivalently, two orbits are either identical or disjoint. See Biggs. The orbits of $G$ partition $X$.

**Orbit-Stabilizer Lemma.** Let $G$ be a finite group of permutations of a set $X$. For $x \in X$, let $O_x = \{\sigma(x) : \sigma \in G\}$ be the orbit of $x$ under $G$ and let $G_x = \{\sigma \in G : \sigma(x) = x\}$ be the stabilizer of $x$ in $G$. Then
$$|G| = |O_x||G_x|.$$  

The proof is postponed.

**Action of permutations on various objects**

A permutation $\sigma$ of a set $X$ can be thought of as permuting subsets of $X$, and sets of subsets of $X$, or mappings from $X$ to a set $C$, etc.

For example, let $\sigma$ be a permutation of $X = \{1, 2, 3, 4, 5\}$. Then $\sigma$ permutes the 10 2-subsets of $X$ by the rule
$$\sigma(\{x, y\}) = \{\sigma(x), \sigma(y)\}.$$  

Perhaps we should use a different symbol for this mapping, like $\overline{\sigma}$ or $\sigma'$; it is not the same as $\sigma$. But we often don’t bother, and this can sometimes be confusing.

For example, if $\sigma$ has cycle decomposition $(1 \ 2 \ 3)(4 \ 5)$ as a permutation of $X$, then the “action” of $\sigma$ on the 2-subsets of $X$ has cycle decomposition
$$\left( \begin{array}{ccc} 1,2 \end{array} \right) \left( \begin{array}{ccc} 1,4 \end{array} \right) \left( \begin{array}{ccc} 2,3 \end{array} \right) \left( \begin{array}{ccc} 3,4 \end{array} \right) \left( \begin{array}{ccc} 4,5 \end{array} \right) \left( \begin{array}{ccc} 5,1 \end{array} \right);$$  

that is, it has one cycle of length 3, one of length 6, and one fixed point. The permutation $\sigma$ also “induces” a permutation of the graphs with vertex set $X$. If the graph $G$ is the pentagon with edges
$$\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}$$
then by $\sigma(G)$ we will mean the graph with edges
$$\{2,3\}, \{3,1\}, \{1,5\}, \{5,4\}, \{4,2\}.$$
This is another pentagon. The cycle decomposition of $\sigma$ as a permutation of all 1024 graphs with vertex set $X$ is very complex.

If $f : X \rightarrow \{R, B\}$ is the mapping

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
R & B & B & R & R
\end{pmatrix}
$$

and $\tau$ is any permutation of $X$, then we can apply $\tau$ to $f$ to get

$$
\begin{pmatrix}
\tau(1) & \tau(2) & \tau(3) & \tau(4) & \tau(5) \\
R & B & B & R & R
\end{pmatrix}.
$$

Here we choose to modify the notation and denote the permutation of mappings by $\hat{\tau}$. So if $\sigma$ is as above, then

$$
\hat{\sigma}(f) = \begin{pmatrix}
2 & 3 & 1 & 5 & 4 \\
R & B & B & R & R
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
B & R & B & R & R
\end{pmatrix}.
$$

Perhaps surprisingly, when we write what we mean out precisely, it is

$$(\hat{\sigma}(f))(i) = f(\sigma^{-1}(i)).$$