Remarks on Maxflow-Mincut

**Proposition 1.** An unaugmentable feasible flow $f$ (i.e. one for which there are no $f$-augmenting paths from $s$ to $t$) is a maxflow (a maximum strength feasible flow).

*Proof:* Review the proof of Maxflow-Mincut to understand; details omitted here. $\square$

There is a gap in our proof of the Mincut-Maxflow Theorem. We started with a maxflow, but we must be sure that such maxflows exists.

One way to see there are maximum strength feasible flows is to note that the flows form a compact set in some Euclidean space, and that “strength” is a continuous function on this set. The set of flows is bounded because of the constraints $0 \leq f(e) \leq c(e)$, and closed because we have $\leq$ in these constraints (and equality in some linear equations) rather than $<$.  

In the case that all capacities are integers, we can see there exist unaugmentable flows in another way. If we start with the 0-flow and use augmenting paths as described, the sequence of flows we get are all integer-valued, and the strength increases by at least 1 each step. Finally we reach an unaugmentable integer flow. As a consequence, we have the following result.

**Theorem 2.** Given a network with digraph $D$, distinguished vertices $s, t$ and a capacity-function that takes nonnegative integer values, there exists an integer-valued maxflow.

It can be shown (we do not do so in this course) that if $D$ has $n$ vertices, and shortest $f$-augmenting paths are used, then after $O(n^3)$ augmentations, we reach an anaugmentable flow.

We partially explained in class how the Mincut-Maxflow theorem implies König’s Theorem on bipartite graphs. Further discussion is omitted from these notes.

**Application to matrices**

Given positive integers $m$ and $n$, we construct a digraph $D_{mn}$ as follows. The vertices are a source $s$, a sink $t$, vertices $x_1, x_2, \ldots, x_m$ and vertices $y_1, y_2, \ldots, y_n$. (These are assumed to be distinct.) The edges are

I $\quad (s, x_i), \ i = 1, 2, \ldots, m,$

II $\quad (y_j, t), \ j = 1, 2, \ldots, n,$ and

III $\quad (x_i, y_j), \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.$

Given an $m \times n$ matrix $A = (a_{ij})$ of real numbers, we get a flow $f_A$ on $D_{m,n}$ by defining

I $\quad f_A(s, x_i) = \sum_{j=1}^{n} a_{ij},$ for $i = 1, 2, \ldots, m,$

II $\quad f_A(y_j, t) = \sum_{i=1}^{m} a_{ij},$ for $j = 1, 2, \ldots, n,$
III  \( f_A(x_i, y_j) = a_{ij} \), for \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \).

Check that \( f_A \) is a flow from \( s \) to \( t \) of strength \( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \), the sum of all entries of \( A \). Every flow from \( s \) to \( T \) in \( D_{mn} \) is of the form \( f_A \) for some matrix \( A \). As an example, here is a \( 2 \times 3 \) matrix \( A \) and the corresponding flow \( F_A \) on \( D_{23} \).

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\]

**Theorem 3.** Let \( A = (a_{ij}) \) be an \( n \times n \) doubly stochastic matrix, i.e. a nonnegative real matrix, all of whose rows and columns sum to 1 (in other words, the row-vectors and column-vectors are probability vectors). Then there exists an \( n \times n \) matrix \( B = b_{ij} \) of 0’s and 1’s all of whose rows and columns sum to 1 (i.e. with a single 1 in each row and each column) so that the positions of the 1’s are chosen from the positions where \( A \) is positive (i.e. if \( b_{ij} = 1 \), then \( a_{ij} > 0 \)).

**Proof:** Make \( D_{nn} \) into a network by giving ALL edges capacity 1. Then \( f_A \) is a feasible flow on this network of strength \( n \). By Theorem 2, there is an integer maxflow (of strength \( n \), and this maxflow is of course of the form \( f_B \) for some integer matrix \( B \). This matrix \( B \) has the required properties. (Details omitted.) \( \square \)

Theorem 3 is an important step in one proof of Birkhoff-von Neumann Theorem on doubly stochastic matrices. See the web.