1. **[NO COLLABORATION.]** We have seen in class that the number of binary words or strings (sequences of 0’s and 1’s) of length \( n \) such that no two ones are consecutive is the Fibonacci number \( f_{n+2} \). (For example, the binary words of length 3 with the property are 000, 100, 010, 001, and 101.)

What is the number of binary words of length \( n \) with exactly \( r \) ones such that no two ones are consecutive? Explain. (This is a combinatorial interpretation of the identity (*) on Notes #14.)

[Don’t use power series; just go back to the ideas of Notes #5.]

2. (i) Show that if \( a_n \) is a polynomial in \( n \) of degree \( d \) (for example, \( a_n = n^2 - 2n + 5 \), where \( d = 2 \)), then the generating function \( f = a_0 + a_1 x + a_2 x^2 + \ldots \) has the form
\[
\frac{p(x)}{(1 - x)^{d+1}}
\]
where \( p(x) \) has degree \( \leq d \).

(ii) Prove the converse: If the generating function \( g = b_0 + b_1 + \ldots \) has the form \( \frac{p(x)}{(1 - x)^{d+1}} \) with \( p(x) \) of degree \( \leq d \), then \( b_n \) is a polynomial of degree \( \leq d \) in \( n \). [Use partial fractions.]

3. If \( f \) is the generating function of a sequence \( a_0, a_1, a_2 \ldots \), what are the generating functions of the following sequences?
   (i) \( a_0 + 5, a_1 + 5, a_2 + 5, \ldots \),
   (ii) \( a_0 + 0, a_1 + 1, a_2 + 2, \ldots \),
   (iii) \( a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots \).

4. Recall that if \( a \) and \( b \) are coprime positive integers, then every positive integer \( n \) that is large enough is a nonnegative linear combination of \( a \) and \( b \). Let \( s_n \) be the number of ways to write \( n = ia + jb \) with \( i \) and \( j \) nonnegative integers. [The old version of this Set said “positive”, a typo.]

   (i) Let \( S(x) = s_0 + s_1 x + s_2 x^2 + \ldots \) be the generating function of the sequence \( s_0, s_1, s_2, \ldots \). What is \( S(x) \)? (I want you to give and briefly justify and answer of the form \( p(x)/q(x) \) where \( p \) and \( q \) are polynomials in \( x \).)

   (ii) Describe a linear recurrence relation that the sequence satisfies.

   (iii) Factor \( q(x) \) into polynomials of the form \( 1 + \alpha x \). (You will need complex roots of unity.)

   (iv) I don’t expect you to find the partial fraction expansion of \( p(x)/q(x) \) completely, but find the coefficients of
\[
\frac{1}{(1 - x)^2} \quad \text{and} \quad \frac{1}{1 - x}.
\]

   (v) Using the partial fraction expansion (which you don’t know completely), estimate \( s_n \). All I want to know is \( \lim_{n \to \infty} s_n/n \).