

The exercises are taken from the text, *Abstract Algebra* (third edition) by Dummit and Foote.

Page 185, 7. (a) $56 = 2^3 \cdot 7$, so if G is abelian of order 56, then $G \cong H \times Z_7$, where H is of order 8. The only possibilities for H are Z_{2^3} , $Z_{2^2} \times Z_2$, and $Z_2 \times Z_2 \times Z_2$, and there are three abelian groups of order 56.

(b) The number of Sylow 7-subgroups must be either 1 or 8. If there are 8 of them, we have 48 distinct elements, and the rest of $56 - 48 = 8$ elements must be all in one Sylow 2-subgroup, which must be normal.

(c) Let K be the normal Sylow 7-subgroup. Consider $\varphi : S \rightarrow \text{Aut}(K)$. Since $\text{Aut}(K) \cong Z_6$, and $|S| = 8$, we must have $|\text{Im}\varphi| = 2$, with $|\ker\varphi| = 4$. For any $s_i \notin \ker$ with $i = 1, 2, 3, 4$, g a generator of K , we have $s \cdot g = g^{-1}$. Suppose $S = Z_2 \times Z_2 \times Z_2$, then $\ker \cong Z_2 \times Z_2$. $G = \{s_1, s_2, s_3, g : g^7 = 1, s_i^2 = 1, s_i g s_i^{-1} = g^{-1}, s_i s_j = s_j s_i\}$.

Suppose $S = Z_4 \times Z_2$. Then $\ker\varphi \cong Z_4$ or $Z_2 \times Z_2$. $G = \{s_1, s_2, g : s_1^4 = 1, s_2^2 = 1, g^7 = 1, s_1 g s_1^{-1} = g, s_2 g s_2^{-1} = g^{-1}\}$ or $\{s_1, s_2, g : s_1^4 = 1, s_2^2 = 1, g^7 = 1, s_1 g s_1^{-1} = k^{-1}, s_2 g s_2^{-1} = k\}$.

Suppose $S = Z_8$, then $\ker \cong Z_4$. $G = \{s, g : s^8 = 1, g^7 = 1, s g s^{-1} = g^{-1}\}$.

Suppose $S = Q_8$. If φ is trivial, we get $Q_8 \times S$, which is nonabelian. If φ is nontrivial, there are 3 subgroups isomorphic to Z_4 , two of which must intersect the kernel nontrivially. $G \cong \{i, j, k, g : i^2 = j^2 = k^2 = -1 = ijk, g^7 = 1, igi^{-1} = jgj^{-1} = g^{-1}, k g k^{-1} = g\}$

Suppose $S = D_8$, φ is determined by r and s . If $\ker\varphi$ is trivial, then $r \mapsto 0$, and $s \mapsto 0$, and $G = D_8 \times S$, which is nonabelian. If $\ker\varphi = \langle r \rangle \cong Z_4$, then $\varphi(r) = 0$, and $\varphi(s) = 3$. If $\ker\varphi = \langle r^2, s \rangle$ or $\langle r^2, sr \rangle \cong Z_2 \times Z_2$. We have $\varphi(r) = 3, \varphi(s) = 0$ or 3 . G induced by these two cases are isomorphic.

(d) Let $g \in G$ be an element of order 7. Note that S is unique. Consider $\varphi : H = \langle g \rangle \rightarrow \text{Aut}(S)$ via conjugation. If φ is trivial, then $G = HS = H \times S$, and H would be a normal Sylow 7-subgroup. Contradiction. Therefore φ is nontrivial, and $\text{Im}\varphi$ has order 7. Since $S \setminus \{1\}$ is of order 7 and has at least 1 element s of order 2, the orbit of s under conjugation by elements of H must be all of $S \setminus \{1\}$. Therefore every nonidentity element in the Sylow 2-subgroup has the same order, which is 2. Hence $S \cong Z_2 \times Z_2 \times Z_2$.

(e) Let $S = Z_2 \times Z_2 \times Z_2$. Then $\text{Aut}(S) \cong GL_3(F_2)$, which has order $(2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 7 \cdot 6 \cdot 4 = 168$. Let P be a Sylow 7-subgroup in $\text{Aut}(S)$. We obtain $S \rtimes_i P$ via the inclusion $i : P \rightarrow \text{Aut}(S)$. If there is another group of order 56 with a nonnormal Sylow 7-subgroup Q , and $\varphi : Q \rightarrow \text{Aut}(S)$, $\text{Im}\varphi$ is again a Sylow 7-subgroup of $\text{Aut}(S)$, which must be a conjugate of P . Therefore $S \rtimes_\varphi Q \cong S \rtimes_i P$. Hence there is a unique group of order 56 with a nonnormal Sylow 7-subgroup.

Page 187, **20**. We will prove the stronger statement:

For all P noncyclic, with $|P| = p^n$, there exists $U \triangleleft P$ such that $U \cong Z_p \times Z_p$ and $K \triangleleft U$ such that $K \triangleleft P$, and $|K| = p$.

If P is of order p^2 , then P is abelian. Since P is noncyclic, $P \cong Z_p \times Z_p$, P obviously satisfies the statement. Suppose P has a normal subgroup $U \cong Z_p \times Z_p$ for all $|P| = p^k$, with $2 \leq k \leq n$. Let P be a group of order p^{n+1} . Since P is a p -group, $Z(P)$ has order divisible by p . Let Z be a subgroup of order p in $Z(P)$, $\bar{P} = P/Z$. If \bar{P} is cyclic, then P is abelian. Since P is not cyclic, by the Fundamental Theorem of F.G. Abelian Groups, P has normal subgroups U and K as in the statement. If \bar{P} is noncyclic, since \bar{P} has order p^n , it has a normal subgroup $\bar{H} \cong U$ by the induction hypothesis. Let H be the complete preimage of \bar{H} in P , then H is noncyclic of order p^3 and normal in P . Let $H_0 = \{x \in H \mid x^p = 1\}$, which is the kernel of the p th power map from H to $Z(H)$, so $|H_0| = p^2$ or p^3 . If $|H_0| = p^2$, then since no element is of order p^2 , $H_0 \cong Z_p \times Z_p$, and easily seen to be normal in P . If $|H_0| = p^3$, then $H = H_0$. Suppose $\bar{K} \triangleleft \bar{H} \triangleleft \bar{P}$, $|\bar{K}| = p$. Let K be the pullback of \bar{K} . Then $|K| = p^2$, and $K \triangleleft H_0$. We must have $K \cong Z_p \times Z_p$. In this case, $Z \triangleleft P$ is the desired subgroup in H_0 of order p .

Page 198, **4**. If G is finite nilpotent, then $G \cong P_1 \times P_2 \times \cdots \times P_s$, where $P_i \in \text{Syl}_{p_i}(G)$ with all possible distinct p_i dividing the order of G . Since each p -group of order p^a has normal subgroups of order p^b for all $b \leq a$, and maximal subgroups M of G are normal, we must have $[G : M]$ is prime.

Page 198, **16**. Suppose H is a maximal subgroup of \mathbb{Q} . Since \mathbb{Q} is divisible, so is \mathbb{Q}/H . Take $x \in \mathbb{Q} \setminus H$, if $\langle \bar{x} \rangle \neq \mathbb{Q}/H$, then $H < \langle H, x \rangle < \mathbb{Q}$. Contradiction. We must have $\langle \bar{x} \rangle = \mathbb{Q}/H \cong Z$ or Z_n for some n , which is not divisible. Contradiction. Therefore, there is no maximal subgroup in \mathbb{Q} .

Page 220, 4. Take any $g \in F(S)$. The quotient of a free group is a free subgroup, so the abelianization of $F(S)$ is a free abelian group isomorphic to a direct sum of Z . By examining the image of the subgroup $\langle g \rangle$ under the quotient, the subgroup generated by g must have infinite order.