MODULAR FORMS AND CALABI-YAU VARIETIES

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Introduction

Let $f(z) = \sum_{n=1}^{\infty} a_n q^n$ be a holomorphic newform of weight $k \geq 2$ relative to $\Gamma_1(N)$ acting on the upper half plane $\mathcal{H}$. Suppose the coefficients a_n are all rational. When $k = 2$, a celebrated theorem of Shimura asserts that there corresponds an elliptic curve E over $\mathbb{Q}$ such that for all primes $p \nmid N$, $a_p = p + 1 - |E(\mathbb{F}_p)|$. Equivalently, there is, for every prime ℓ, an ℓ-adic representation ρ_ℓ of the absolute Galois group $\mathfrak{G}_\mathbb{Q}$ of $\mathbb{Q}$, given by its action on the ℓ-adic Tate module of E, such that a_p is, for any $p \nmid \ell N$, the trace of the Frobenius Fr_p at p on ρ_ℓ. Denote by $t = t(N)$ the order of the torsion subgroup of $\Gamma_1(N)$.

The primary aim of this article is to provide some positive evidence for the expectation that, for every $k \geq 2$, any rational newform f of weight k and level N should have an associated Calabi-Yau variety $X/\mathbb{Q}$ of dimension $k-1$ such that

(Ai) The $\{(k-1,0),(0,k-1)\}$-piece of $H^{k-1}(X)$ splits off as a submotive M_f,

(Aii) $a_p = \mathrm{tr}(Fr_p \,|\, M_{f,\ell})$, for almost all p, and

(Aiii) $\det(M_{f,\ell}) = \chi_\ell^{k-1}$,

where χ_ℓ is the ℓ-adic cyclotomic character. We will focus on the even weight case for small levels.

One would in fact want (Aii) to hold for any $p \nmid \ell t N$. For simplicity, one could think of a motive as a semisimple motive M relative to absolute Hodge cycles ([DMOS]), with realizations $(M_B, M_{\mathrm{dR}}, M_\ell)$. Since every Calabi-Yau manifold of dimension 1 is an elliptic curve, (C1) is a natural extension of what one has for $k = 2$. Of course for $k > 2$, one knows by Deligne ([Del]) that there is an irreducible, 2-dimensional ℓ-adic representation ρ_ℓ so that $(Aii), (Aiii)$ hold with $M_{f,\ell}$ replaced by ρ_ℓ.

When the weight is odd, the $\mathbb{Q}$-rationality forces f to be of CM type, and for weight 3, we refer to [ES] for a beautiful recent result.

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For non-CM forms f of weight $k > 2$ with $\mathbb{Q}$-coefficients, we in fact hope for more, namely that the cohomology ring of X will be spanned by M and the Hodge/Tate classes of various degrees; in particular, X should be rigid in this case.

It is a difficult problem in dimensions > 3 to find smooth models of varieties with trivial canonical bundles, and for this reason we formulate our questions for such varieties with mild singularities. By a *Calabi-Yau variety* over a field k, we will mean an n-dimensional projective variety X/k on which the canonical bundle $\mathcal{K}_X$ is defined such that

(CY1) $\mathcal{K}_X$ is trivial; and
(CY2) $H^m(X, \mathcal{O}_X) = 0$ for all (strictly) positive $m < n$.

More precisely, we will want such an X to be normal and Cohen-Macaulay, so that the dualizing sheaf $\mathcal{K}_X$ is defined, with the singular locus in codimension at least 2, so that $\mathcal{K}_X$ defines a Weil divisor; finally, X should be Gorenstein, so that $\mathcal{K}_X$ will represent a Cartier divisor. Ideally we would like to singular locus X_{sing} to be of dimension $\leq \left[\frac{n-1}{2}\right]$.

In addition to these properties, we would ideally also like X to be realized as a double cover $\pi : X \to Y$, where Y is a projective smooth rational variety with negative ample canonical divisor. (Such an X will be automatically Gorenstein so that $\mathcal{K}_X$ is defined, and $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y \oplus L$, for a line bundle L on Y with L^2 giving the branch locus; moreover, $\pi_*(\mathcal{K}_X) = \mathcal{K}_Y \oplus \mathcal{K}_Y \otimes L^{-1}$, forcing $L = \mathcal{K}_Y$.)

Here is our first result:

Theorem 1 *Fix $\Gamma = \Gamma_1(N)$, $N \leq 5$. Let k be the first even weight s.t. dim$(S_k(\Gamma)) = 1$. Then $\exists$ a Calabi-Yau variety $V(f)/\mathbb{Q}$ associated to the new generator f of $S_k(\Gamma)$ satisfying (Ai) and (Aii). In fact, when $N \leq 5$, $V = V(f)$ is birational to the Kuga-Sato variety $\tilde{\mathcal{E}}_N^{(k-2)}$.*

In particular, this result applies to the Delta function $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = \prod_{m \geq 1}(1 - q^m)^{24}$, for $N = 1, k = 12$. By the Kuga variety, we mean a suitable compactification (see section 1) of the fibre product E_N^{k-2} of the universal elliptic curve E_N over the model over $\mathbb{Q}$ of the modular curve $\Gamma_1(N)\backslash\mathcal{H}$.

It is well known that $S_k(\Gamma_1(N))$ is **one-dimensional** when (N, k) equals $(\mathbf{1}, \mathbf{12}), (1, 16), (\mathbf{2}, \mathbf{8}), (2, 10), (\mathbf{3}, \mathbf{6}), (3, 8), (\mathbf{4}, \mathbf{6}), (\mathbf{5}, \mathbf{4}), (5, 6), (\mathbf{6}, \mathbf{4}), (7, 4)$. (The ones in bold are the cases to which the Theorem applies.) For example, for the case $(\mathbf{2}, \mathbf{8})$, the generator is

$$f(z) = q - 8q^2 + 12q^3 - 210q^4 + 1016q^5 + \ldots$$

Recall that a newform $f(z) = \sum_{n=1}^{\infty} a_n q^n$ is of *CM-type* iff there is an odd, quadratic Dirichlet character δ such that $a_p = a_p\delta(p)$ for almost all primes p. Equivalently, if K is the imaginary quadratic field cut out by δ, $a_p = 0$ for almost all p which are inert in K.

As a first step, we may ask for a *potential statement*, i.e., the association, over a finite extension k of $\mathbb{Q}$, of a Calabi-Yau variety V/k to a newform f with $\mathbb{Q}$-coefficients. Here is a modest result in this direction:

Theorem 2 Let f be a newform of weight $k \geq 3$ of CM type with rational coefficients. Then $\exists$ a Calabi-Yau $(k-1)$-fold X defined over a number field k such that $(Ai), (Aii)$ hold over k. This X arises as a Kummer variety associated to an elliptic curve E with complex multiplication. When $k \leq 4$, X can be taken to be a smooth model.

Again, when $k = 3$, there is a much more precise and satisfactory result over $\mathbb{Q}$ in [ES].

In the converse direction, if M is a simple motive over $\mathbb{Q}$ of rank 2, with coefficients in $\mathbb{Q}$, of Hodge type $\{(w, 0), (0, w)\}$ with $w > 0$, then the general philosophy of Langlands, and also a conjecture of Serre, predicts that M should be modular and be associated to a newform f of weight $w + 1$ with rational coefficients. This is part of a very general phenomenon, and applies to motives occurring in the cohomology smooth projective varieties over $\mathbb{Q}$. In any case, it applies in particular to Calabi-Yau threefolds over $\mathbb{Q}$ whose $\{(3, 0), (0, 3)\}$-part splits off as a submotive. In this context, there have been a number of beautiful results, some of which have been described in the monographs [YL] and [YYL]. They are entirely consistent with what we are trying to do in the opposite direction, and also provide supporting examples. It should perhaps be remarked that these results (relating to the modularity of rigid Calabi-Yau threefolds) can now be deduced *en masse* from the recent proof of Serre's conjecture due to Khare and Wintenberger, with a key input from Kisin.

Let us now move to a more general situation. Fix any positive integer n and suppose that f is a (new) Hecke eigencusp form on the symmetric space

$$\mathcal{D}_n := \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n),$$

relative to a congruence subgroup Γ of $\mathrm{SL}(n, \mathbb{Z})$, which is *algebraic* and *regular*. For $n = 2$, f is algebraic and regular iff it is holomorphic of weight ≥ 2. In general, one considers the cuspidal automorphic representation π of $\mathrm{GL}(n, \mathbb{A})$ which is generated by f, and by Langlands the archimedean component π_∞ corresponds to an n-dimensional representation σ_∞ of the real Weil group $W_\mathbb{R}$, which contains $\mathbb{C}^*$ as a subgroup of index 2. One says ([Cℓ]) that π is algebraic if the restriction of σ_∞ to $\mathbb{C}^*$ is a sum of characters χ_j of the form $z \to z^{p_j}\overline{z}^{q_j}$, with $p_j, q_j \in \mathbb{Z}$, and it is regular iff $\chi_i \neq \chi_j$ when $i \neq j$. Such an f contributes to the *cuspidal cohomology* $H^*_{\rm cusp}(\Gamma\backslash\mathcal{D}_n, V)$ relative to a *local coefficient system* V in a specific degree $w = w(f)$. Moreover, f is rational over a number field $\mathbb{Q}(f)$, defined by the Hecke action on cohomology, which preserves the cuspidal part (*loc. cit.*). There is conjecturally a motive $M(f)$ over $\mathbb{Q}$ of rank n, with coefficients in $\mathbb{Q}(f)$, and weight w. By Clozel [Clo], $M_\ell(f)$ exists for suitable f which are in addition *essentially selfdual.*

Now let f be an algebraic, regular, essentially selfdual newform of weight w relative to $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$, with L-function $L(s,f) = \prod_p L_p(s,f)$, such that $\mathbb{Q}(f) = \mathbb{Q}$. Then our question is if there exists a Calabi-Yau variety $X/\mathbb{Q}$ of dimension w such that

(Ci) There is a submotive $M(f)$ of $H^w(X)$ of rank n such that $M(f)^{(w,0)} = H^{w,0}(X)$, and

(Cii) $L_p(s,f) = L_p(s, M_\ell(f))$ for almost all p,

where $L_p(s, M_\ell(f))$ equals, at any prime $p \neq \ell$ where the ℓ-adic realization $M_\ell(f)$ is unramified, $\det(I - Fr_p p^{-s} \,|\, M_\ell(f))^{-1}$. Again we would like to be able to find an X having good reduction outside the primes dividing tN, where n is the level of f and t the order of torsion in Γ, such that (Cii) holds for any such $p \neq \ell$.

Thanks to the *principle of functoriality*, one should be able to obtain a certain class of $\mathbb{Q}$-rational, regular, algebraic, essentially selfdual newforms f by transferring forms on (the symmetric domains of) smaller reductive $\mathbb{Q}$-subgroups G of $\mathrm{GL}(n)$. The simplest instance of this phenomenon is given by the symmetric powers $\mathrm{sym}^m(g)$ of classical $\mathbb{Q}$-rational, non-CM newforms g of weight k. One knows by Kim and Shahidi ([KS], [Kim]) that for $m \leq 4$, $f = \mathrm{sym}^m(g)$ is a cusp form on $\mathrm{GL}(m+1)$. Here is our third result:

Theorem 3 *Let g be a non-CM, elliptic modular newform of weight 2, level N and trivial character, whose coefficients a_n lie in $\mathbb{Q}$. Then for for any $m > 0$, there is a Calabi-Yau variety X_m over $\mathbb{Q}$ of dimension m associated to f such that $(Ci), (Cii)$ hold. Moreover, for $m \leq 3$, X_m can be taken to be non-singular, with good reduction outside $m!N$.*

Here X_2 is just the familiar Kummer surface attached to $E \times E$, where $E = X_1$ is the elliptic curve/$\mathbb{Q}$ defined by g. But the case $m = 3$ is interesting, especially since it is not rigid, thanks to the Hodge type being $\{(3,0),(2,1),(1,2),(0,3)\}$, with each Hodge piece being one-dimensional. In fact, in that case, $\mathrm{sym}^3(g)$ corresponds (by [RS]) to a holomorphic Siegel modular cusp form F of genus 2 and (Siegel) weight 3. Such an F contributes to the cohomology in degree 3 of the Siegel modular threefold V of level N^3. Since the geometric genus of V is typically > 1, it cannot be Calabi-Yau. However, there should be, as predicted by the Hodge and Tate conjectures, an algebraic correspondence between X_3 and V (for any N).

One also knows (cf. [Ram]) that given two non-CM newforms g, h of weights $k, r \geq 2$ respectively, then there is an algebraic automorphic form $f = g \boxtimes h$ on $\mathrm{GL}(4)/\mathbb{Q}$, which will be cuspidal and regular if $k \neq r$. If g, h are $\mathbb{Q}$-rational, then so is f. Moreover, f is essentially selfdual because g and h are.

Theorem 4 *Let g, h be $\mathbb{Q}$-rational, non-CM newforms as above of respective weights k, r, with $k \neq r$. Suppose we have Calabi-Yau varieties $X(g), X(h)$ attached to g, h, satisfying $(Ai), (Aii), (Aiii)$. Put $f = g \boxtimes h$, so that $w(f) = (k-1)(r-1)$. Then there is a Calabi-Yau variety $X(f)/\mathbb{Q}$ of dimension $w(f)$ such that $(Ci), (Cii)$ hold.*

The point is that the product $Z := X(g) \times X(h)$ has the desired submotive in degree w, but it has global holomorphic m-forms for $m = k-1$ and $m = r-1$. We exhibit an involution τ on Z such that when we take the quotient by τ, these forms get killed and we get a Calabi-Yau variety with reasonable singularities. To get unconditional examples of this Theorem, take $k = 2$ and choose h to be one of the examples of Theorem A of weight $r > 2$. To be specific, we may take g to be the newform of weight 2 and level 11, and h to be the newform of weight 4 and level 5, in which case $f = g \boxtimes h$ has level 55^2, and $X(f)$ is a Calabi-Yau fourfold.

In sum, it is a natural question, given Shimura's work on for of weight 2, to hope for Calabi-Yau varieties associated to forms of higher weight with rational coefficients, and the first author raised this question in a talk at the Borel memorial conference at Zhejiang University in Hangzhou, China, in 2004. Quite appropriately, Dick Gross, who was in the audience, cautioned against hoping for too much without sufficient evidence. Over the past years, there has been some positive evidence, though small, and even if there is no V in general, especially for non-CM forms f of even weight, the examples where one has nice Calabi-Yau varieties V enriches them considerably, and one of our aims is to understand the Hecke eigenvalues a_p in such cases a bit better in terms of counting points of V mod p. Recently we have learnt from [ES] (and Noriko Yui) that the question of existence of a Calabi-Yau variety

V associated to f has also been voiced by Mazur and van Stratten. What we truly hope for is that in addition, V will be equipped with an involution, so one can form products, etc., and also deal with quadratic twists. We have some interesting examples in the 3-dimensional case (where $k = 4$), and an expanded version of this paper will also contain a discussion of these matters. In a sequel, we will discuss a way to get nicer models in certain higher dimensional examples. It should also be noted that, in a different direction, (families of) Calabi-Yau varieties of Dwork type have bee employed in the spectacular works of Clozel, Harris, Shepherd-Barron and Taylor (see [HSBT]) in their proof of potential modularity of symmetric powers of elliptic curves with multiplicative reduction.

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1. A geometric construction

Our object is to give a construction of varieties with trivial canonical bundles arising as birational models of elliptic modular varieties V with non-positive canonical bundles. Recall that V arise as fibre products of the universal family of elliptic curves E with additional structures over the modular curve associated to a congruence subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$. When Γ is $\Gamma_1(N)$ (resp. $\Gamma_0(N)$), the additional structure is a point (resp. subgroup) of order N. We have a slight preference for $\Gamma_1(N)$ over $\Gamma_0(N)$, because in the latter case, one gets, due to the existence of $-I$, only a coarse moduli space.

Let us consider a *general moduli problem* of the type $(E, S; R)$ where E is a curve of genus 1, S a finite set of points on E, and R a finite set of linear equivalence relations on the points S. Suppose that there is a surface X, and a linear system P of divisors linear equivalent to $-K_X$, where K_X is the canonical divisor of X. Assume that for a "general" datum $(E, S; R)$ we have a uniquely determined element of P, so that the projective space P is birationally the moduli space of the above moduli problem.

Let $W \subset \Gamma(X, \mathcal{O}(-K_X))$ be the subspace so that P is the associated projective space and $n = \mathrm{rank} W$. We have a natural homomorphism

$$\phi : V \otimes \mathcal{O}_{X^n} \to \oplus_{i=1}^n p_i^* \mathcal{O}(-K_X)$$

The divisor D where the determinant $\det(\phi)$ vanishes is birationally the moduli space for the moduli problem of the type $(E, S : R, p_1, \ldots, p_n)$ where p_i are n additional points which are not subject to any additional relation. Moreover, D has trivial canonical bundle.

In the context of the congruence subgroup $\Gamma_1(N)$ of $\mathrm{SL}(2,\mathbb{Z})$, the varieties V with non-positive canonical bundles are associated with pairs (N,k) such that there is at most one modular form of level N and weight k. The complete list of such pairs is given in the following table:

N	k
1	≤ 23, 25, 26, odd
2	≤ 11, odd
3	≤ 8
4	≤ 6
5, 6	≤ 4
7, 8	≤ 3
9, 10, 11, 12, 14, 15	2

One can similarly make the corresponding tables for the groups $\Gamma_0(N)$ and $\Gamma_0(N^2)\cap\Gamma_1(N)$.

2. The Calabi-Yau 11-fold associated to Δ

Δ is a generator of $S_{12}(\mathrm{SL}(2,\mathbb{Z}))$. The object is to show that the Kuga-Sato variety $\mathcal{E}^{(10)}$ is birational to an eleven-dimensional Calabi-Yau variety V. We will use $\equiv$ to denote birational equivalence. It is easy to see that for any $r \geq 0$,

$$\mathcal{E}^r \;\equiv\; \mathcal{M}_1(r+1),$$

where $\mathcal{M}_g(k)$ is the *moduli space genus g curves with k marked points*, with compactification $\overline{\mathcal{M}}_g(k)$.

So we need to find a *birational model* V of $\overline{\mathcal{M}}_1(11)$ such that V is Calabi-Yau. Let S be the surface obtained by *blowing up* 4 *general points* P_1,P_2,P_3,P_4 in $\mathbb{P}^2$. Let E be an elliptic curve with $n+5$ general points $Q_0,Q_1,\ldots,Q_{n+4}$, and use $|3Q_0|$ to define a morphism $E\to\mathbb{P}^2$.

Using an automorphism of $\mathbb{P}^2$ we may assume: $Q_i = P_i$ for $1\leq i\leq 4$. The embedding $E\to\mathbb{P}^2$ lifts to a morphism $\varphi : E\to S$, and the *adjunction formula* gives $\varphi(E)\in|\mathcal{K}_S^{-1}|$.

Get a *rational map* $\mathcal{M}_1(n+5)\to S^n$,

$$(E,\{Q_0,\ldots,Q_{n+4}\})\to(P_5,\ldots,P_{n+4}).$$

$W := \Gamma(S,\mathcal{K}_S^{-1})$ has dim. 6, and $\exists$ a hom (of sheaves on S^n):

$$f_n : W\otimes\mathcal{O}_{S^n} \;\to\; \mathcal{K}_S^{-1}\boxtimes\cdots\boxtimes\mathcal{K}_S^{-1}.$$

$\mathrm{Ker}(f_n)$ is the vector space of *sections in* W *vanishing at* $= P_5, \ldots, P_{n+4}$. The associated projective space then identifies with the collection of all (general) points in $\mathcal{M}_1(11)$ giving rise to this point on S^n.

Put

$$V_n := \mathrm{Proj}_{S^n}(\mathrm{coker}({}^t f_n))$$

where

$${}^t f_n : \mathcal{K}_S \boxtimes \cdots \boxtimes \mathcal{K}_S \to W^\vee \otimes \mathcal{O}_S^\vee.$$

Then V_n is birational to $\mathcal{M}_1(n+5)$. We have $\mathrm{corank}({}^t f_n) = 6 - n$ at a general point of S^n. And there is a natural map

$$\pi : V_n \to S^n$$

$\mathbf{n \leq 5}$: π is surjective with fibres $\mathbb{P}^{5-n}$. Hence V_n, which is $\equiv \overline{\mathcal{M}}_1(n+5)$, is a *rational variety* in this case.

$\mathbf{n = 6}$: $V = V_6$ is a *divisor in* S^6 defined by the vanishing of $\det(f_n)$, which is a section of $\mathcal{K}_{S^n}^{-1}$. So $\mathcal{K}_V$ **is trivial**. We already know that $h^{(11,0)} = 1$ and $h^{(p,0)} = 0$ for $0 < p < 11$ for $\tilde{\mathcal{E}}^{10}$. These also hold for V. So V is Calabi-Yau. Also, the whole construction is rationally defined.

□

3. A C-Y 7-fold occurring in level 2

Let E be an elliptic curve with origin $o \in E$, $x \in E$ a point of order 2 and $y, z \in E$ some other (general) points. Under the morphism $a : E \to |2[o] + [y]|$, the divisors $2[o] + [y]$ and $2[x] + [y]$ are linear sections. Of the four points o, x, y and z we may assume (under the hypothesis of generality on y and z) that no three are collinear. Thus, we can identify these with the points $(0:0:1)$, $(1:0:0)$, $(0:1:0)$ and $(1:1:1)$ respectively in order to identify $|2[o]+[y]|$ with $\mathbb{P}^2$.

Conversely, let E be a cubic curve in $\mathbb{P}^2$ which has the following properties:

(1) E passes through the points $(0:0:1)$, $(0:1:0)$, $(1:0:0)$ and $(1:1:1)$.
(2) The line $Y = 0$ is tangent to E at the point $(0:0:1)$.
(3) The line $Z = 0$ is tangent to E at the point $(0:1:0)$.

Then E is a curve of genus 1 for which we take $o = (0:0:1)$ as the origin of a group law. Let $x = (0:1:0)$, $y = (1:0:0)$ and $z = (1:1:1)$. Then we obtain the relation

$$2[o] + [y] \simeq 2[x] + [y]$$

It follows that $2x = o$. Thus we have obtained (E, o, x, y, z) of the type we started with.

Direct calculation shows that the linear system of cubics in $\mathbb{P}^2$ that satisfy the conditions above is the linear span of $X^2Y - XYZ$, $X^2Z - XYZ$, $Y^2Z - XYZ$ and $YZ^2 - XYZ$.

Here is an **alternate construction in level** 2:

Let E be an elliptic curve with origin $o \in E$, $x \in E$ a point of order 2 and $y, z \in E$ some other points. The morphism $a : E \to |2[o]|$ has fibres $2[o]$, $[y] + [-y]$ and $[z] + [-z]$ which we map to 0, 1 and ∞ respectively in order to identify $|2[o]|$ with $\mathbb{P}^1$. The morphism $b : E \to |[o] + [x]|$ has fibres $[o] + [x]$, $[y] + [x - y]$ and $[z] + [x - z]$ which we map to 0, 1 and ∞ in order to identify $|[o] + [x]|$ with $\mathbb{P}^1$. Thus we obtain a morphism $a \times b : E \to \mathbb{P}^1 \times \mathbb{P}^1$ which is constructed canonically from the data (E, o, x, y, z).

Conversely let E be a curve of type $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ which has the following properties:

(1) E passes through the points $(0, 0)$, $(1, 1)$ and $(\infty, infty)$.
(2) The line $\{0\} \times \mathbb{P}^1$ is tangent to E at the point $(0, 0)$.
(3) If $(u, 0)$ is the residual point of intersection of E with $\mathbb{P}^1 \times \{0\}$, then $\{u\} \times \mathbb{P}^1$ is tangent to E at this point.

Then E is a curve of genus 0 for which we take $o = (0, 0)$ as the origin in a group law. Let $x = (u, 0)$, $y = (1, 1)$ and $z = (\infty, infty)$. We obtain the identities $2[o] \simeq 2[x]$ from which it follows that $2x = o$. Thus we have recovered the data (E, o, x, y, z).

4. Remark on Elliptic curves with Level 3 structure

Let E be an elliptic curve with origin $o \in E$, $x \in E$ a point of order 3 and $y \in E$ some other point. The morphism $a : E \to |2[o]|$ has fibres $2[o]$, $[x] + [2x]$ and $[y] + [-y]$ which we map to 0, 1 and ∞ respectively in order to identify $|2[o]|$ with $\mathbb{P}^1$. The morphism $b : E \to |[o] + [x]|$ has fibres $[o] + [x]$, $2[2x]$ and $[y] + [x - y]$ which we map to 0, 1 and ∞ respectively in order to identify $|[o] + [x]|$ with $\mathbb{P}^1$. Thus we obtain a morphism $a \times b : E \to \mathbb{P}^1 \times \mathbb{P}^1$ which is constructed canonically from the data (E, o, x, y).

Conversely, let E be a curve of type $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ which has the following properties:

(1) E is tangent to the line $\{0\} \times \mathbb{P}^1$ at the point $(0, 0)$.
(2) E is tangent to the line $\mathbb{P}^1 \times \{1\}$ at the point $(1, 1)$.
(3) E passes through the points $(1, 0)$ and $\infty, \infty)$.

Then E is a curve of genus 1 and we use $o = (0, 0)$ as the origin of a group law on E. Let $x = (1, 0)$, $y = (\infty, \infty)$ and $p = (1, 1)$. We obtain the identities,

$$2[o] \simeq [x] + [p] \qquad\qquad [o] + [x] \simeq 2[p]$$

It follows that $p = 2x$ and $3x = o$. We have thus recovered the data (E, o, x, y) that we started with.

Direct calculations shows that the linear system of cubic curves in $\mathbb{P}^2$ satisfying the conditions given above is the linear span of $X^2Y - XYZ$, $Y^2Z - XYZ$ and $Z^2X - XYZ$.

5. Level 3 and a CY 5-fold

We will construct a 5-fold with trivial canonical bundle and singularities only in dimension 2 or less such that its middle cohomology represents the motive of the (unique) modular form of level 3 and weight 6.

Consider the linear system P of cubics in $\mathbb{P}^2$ that is spanned by the curves $X^2Y - XYZ$, $Y^2Z - XYZ$ and $Z^2X - XYZ$; this system is stable under the cyclic automorphism $X \to Y \to Z \to X$ of $\mathbb{P}^2$. Each curve in the linear system P is tangent to the line $Z = 0$ at the point $p_Y = (0 : 1 : 0)$; similarly, the curve is tangent to $X = 0$ at the point $p_Z = (0 : 0 : 1)$ and to $Y = 0$ at the point $p_X = (1 : 0 : 0)$. Moreover, each curve passes through the point $p_0 = (1 : 1 : 1)$. We note that the linear system P is precisely the collection of cubic curves in $\mathbb{P}^2$ that satisfy these conditions.

In the divisor class group of a smooth curve in this linear system we obtain the identities

$$2p_Y + p_X = 2p_Z + p_Y = 2p_X + p_Z = p_X + p_0 + r$$

where r denotes the remaining point of intersection of the curve with the line $Y = Z$ that joins p_X and p_0. In particular, we note that $p_Y - p_X$ is of order 3 in this class group and $p_Z - p_X = 2(p_Y - p_X)$.

Conversely, suppose we are given a smooth curve E of genus 1 and a line bundle ξ of order 3 on E; moreover, suppose that three distinct points p, q and r are marked on E. We then obtain two additional points a and b on E such that $a - p = \xi$ and $b - p = 2\xi$ in the divisor class group of E. Consider the morphism $E \to \mathbb{P}^2$ that is given by the linear system of the divisor $p + q + r$. Moreover, we choose co-ordinates on $\mathbb{P}^2$ so that the point p goes to p_X, q goes to p_0, a goes to p_Y and b goes to p_Z. This gives an embedding of E as a curve in $\mathbb{P}^2$ that belongs to the linear system P.

Let S denote the surface obtained by blowing up $\mathbb{P}^2$ at the four points p_X, p_Y, p_Z and p_0, and then further blowing up the resulting surface at the "infinitely near points" that correspond to $Z = 0$ at p_Y, to $X = 0$ at p_Z and to $Y = 0$ at p_X. Let H denote the inverse image in S of a general line in $\mathbb{P}^2$; let E_X, E_Y, E_Z and E_0 denote the strict transforms of the exceptional loci of the first blow-up over the points p_X, p_Y, p_Z and p_0 respectively; let F_X, F_Y and F_Z denote the exceptional divisors of the second blow-up. The anti-canonical divisor $-K_S = 3H - E_X - E_Y - E_Z - E_0 - 2(F_X - F_Y - F_Z)$ has a base-point free complete linear system $|-K_S|$ which can be identified with P. Let T denote the natural incidence locus in $S \times P$. The variety

$$X = T \times_P T \times_P T$$

is a singular 5-fold which is Gorenstein and has trivial canonical bundle. Moreover, an open subset of X_0 parametrizes tuples of the form $(E, \xi, p, q, r, s, t, u)$ where E is a curve of genus 1, ξ is a line bundle of order 3 on E and p, q, r, s, t and u are six distinct points on E.

Let L_X, L_Y, L_Z denote the strict transforms in S of the lines in $\mathbb{P}^2$ defined by $X = 0$, $Y = 0$, $Z = 0$ respectively. Further, let R be the strict transform in S of the curve in $\mathbb{P}^2$ defined by

$$X^2Z + Y^2X + Z^2Y - 3XYZ = 0$$

This is the unique cubic in $\mathbb{P}^2$ that has a node at p_0 and is tangent to $X = 0$ at p_Y, to $Y = 0$ at p_Z and to $Z = 0$ at p_X. It follows that R is a smooth rational curve that meets E_0 in a pair of distinct points and the triple (R, L_X, E_Y) (respectively (R, L_Y, E_Z) and (R, L_Z, E_X)) consists of smooth curves that meet pairwise transversally.

The morphism $S \to P^*$ induced by the linear system P can be factorized via a double cover $S \to W$ which is ramified along R. Each of the curves L_X and E_Y (respectively L_Y and E_Z; L_Z and E_X) is mapped isomorphically onto the same smooth irreducible curve G_Z (respectively G_X; G_Y) in W; the curve R is mapped isomorphically onto the branch locus Q in W. The morphism $W \to P^*$ collapses the curves G_X (respectively G_Y and G_Z) to a point q_X (respectively q_Y and q_Z) in P^*; in fact $W \to P^*$ is identified with the blow-up of P^* at these points. Moreover, Q is mapped to a plane quartic $\overline{Q}$ which has cusps at these three points.

Let T denote the incidence locus in $P \times S$ as above. It is the pull-back via $S \to P^*$ of the natural incidence locus $I \subset P \times P^*$. The latter can be identified (via the projection $I \to P^*$) with the projective bundle of 1-dimensional linear subspaces of the tangent bundle of P^*. Hence, the exceptional curve G_X (respectively G_Y and G_Z) of the blow-up $W \to P^*$ can be identified with the fibre I_X of $I \to P^*$ over the point q_X (respectively q_Y and q_Z). Thus we obtain natural maps $E_\alpha \to T$ and $L_\alpha \to T$ that are sections of the $\mathbb{P}^1$-bundle $T \to S$ over the curves E_α and L_α respectively; let $\tilde{E_\alpha}$ and $\tilde{L_\alpha}$ denote the images. Let T_X

(respectively T_Y; T_Z) denote the fibre of T over the point of intersection of L_Y and E_Z (respectively L_Z and E_X; L_X and E_Y).

The tangent direction along $\overline{Q}$ gives a rational morphism (defined outside the cusps) from $\overline{Q}$ to I. It follows that this extends to a section $R \to T$ of $T \to S$ over R and gives a curve $\tilde{R}$ in T. The quadruple of curves $\tilde{R}$, T_X, $\tilde{L_Y}$, $\tilde{E_Z}$ (respectively, $\tilde{R}$, T_Y, $\tilde{L_Z}$, $\tilde{E_X}$; $\tilde{R}$, T_Z, $\tilde{L_X}$, $\tilde{E_Y}$) meet pairwise transversally in a single point r_X (respectively r_Y; r_Z) in T. The curve $\tilde{R}$ in T is mapped to a nodal cubic $\overline{R}$ in P for which I_X, I_Y and I_Z are inflectional tangents. The curves T_X, $\tilde{L_Y}$, $\tilde{E_Z}$ (respectively T_Y, $\tilde{L_Z}$, $\tilde{E_X}$; T_Z, $\tilde{L_X}$, $\tilde{E_Y}$) in T lie over I_X (respectively I_Y; I_Z) in P.

The singular locus of the morphism $T \to P$ consists of the curves $\tilde{R}$, T_α, $\tilde{L_\alpha}$ and $\tilde{E_\alpha}$ for $\alpha = X, Y, Z$ as described above.

The singular fibres of $T \to P$ then have the following description:

(1) If a is a smooth point of $\overline{R}$ which is not a point of inflection then the fibre C_a is a rational curve in S with a single ordinary node.
(2) If b which is on an inflectional tangent (i. e. one one of the lines I_X, I_Y, I_Z) of $\overline{R}$ but is *not* a point of inflection of $\overline{R}$ then the fibre C_b is a curve with three components and three nodes (i. e. a "triangle" of $\mathbb{P}^1$'s).
(3) If c is a point of inflection of the curve $\overline{R}$, then C_c consists of three $\mathbb{P}^1$'s that pass through a point and (since C_c lies on a smooth surface S) is locally a complete intersection.
(4) If d is the node of $\overline{R}$ then the fibre C_d consists of a pair of smooth $\mathbb{P}^1$'s in S that meet in a pair of points. In fact the curves are E_0 and the strict transform in S of the curve in $\mathbb{P}^2$ defined by the equation

$$X^2Y + Y^2Z + Z^2X - 3XYZ = 0$$

In particular, the elliptic fibration $T \to P$ is semi-stable but for the three fibres over the points of inflection of $\overline{R}$.

Now consider the variety $X = T \times_P T \times_P T$. The singular points of X_0 consist of triples (x, y, z) of points of T, where at least two of these points are critical points for the morphism $T \to P$. In particular, these points lie over the union of R and I_X, I_Y and I_Z. Since the singular points of each of the fibres described above are isolated, it follows that the singular locus of X has components of dimension at most 2.

6. Level 4 and a C-Y 5-fold

Let E be an elliptic curve with origin $o \in E$, $x \in E$ a point of order 4. Under the morphism $a : E \to |3[o]|$, the divisors $3[o]$, $2[2x] + [o]$, $[2x] + 2[x]$ and $[o] + [x] + [3x]$ are linear sections of the image curve. Let p denote the point of intersection of the lines corresponding to $3[o]$ and $[2x] + 2[x]$. No three of the points o, $2x$, $3x$ and p are collinear. Thus we can identify $|3[o]|$ with $\mathbb{P}^2$ in such a way that o is identified with $(0 : 1 : 0)$, $3x$ is identified with $(0 : 0 : 1)$, p is identified with $(1 : 0 : 0)$ and $3x$ is identified with $(1 : 1 : 1)$. Thus we obtain a morphism $a : E \to \mathbb{P}^2$ which is constructed canonically from the data (E, o, x).

Conversely, let E be a cubic curve in $\mathbb{P}^2$ which has the following properties:

(1) E passes through the points $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(1 : 1 : 1)$ and $(1 : 0 : 1)$.
(2) The line $Z = 0$ is an inflectional tangent to E (at the point $(0 : 1 : 0)$).
(3) The line $X = 0$ is tangential to the curve E at the point $(0 : 0 : 1)$.
(4) the line $Y = 0$ is tangential to the curve E at the point $(1 : 0 : 1)$.

Then E is a curve of genus 0 and we use $o = (0 : 1 : 0)$ as the origin of a group law on E. Let $x = (1 : 0 : 1)$, $p = (0 : 0 : 1)$ and $q = (1 : 1 : 1)$. We obtain the identities

$$3[0] \simeq [o] + 2[p] \qquad \simeq [p] + 2[x] \simeq [o] + [q] + [x]$$

It follows that $2p = o$, $2x = p$ and $q = -x = 3x$. Thus we have recovered the data (E, o, x) that we started with.

Direct calculation shows us that the linear system of cubic curves in $\mathbb{P}^2$ that satisfy the above conditions is the linear span of $YZ(Y - Z)$ and $X(X - Z)^2$.

Here is an **alternate construction in level** 4:

Let E be an elliptic curve with origin $o \in E$, $x \in E$ a point of order 4 and $y \in E$ some other point. The morphism $a : E \to |2[o]|$ has fibres $2[o]$, $2[2x]$ and $[y] + [-y]$ which we map to 0, 1 and ∞ respectively in order to identify $|2[o]|$ with $\mathbb{P}^1$. The morphism $b : E \to |[o] + [x]|$ has fibres $[o] + [x]$, $[2x] + [3x]$ and $[y] + [x - y]$ which we map to 0, 1 and ∞ respectively in order to identify $|[o] + [x]|$ with $\mathbb{P}^1$. Thus we obtain a morphism $a \times b : E \to \mathbb{P}^1 \times \mathbb{P}^1$ which is constructed canonically from the data (E, o, x, y).

Conversely, let E be a curve of type $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ which has the following properties:

(1) E is tangent to the line $\{0\} \times \mathbb{P}^1$ at the point $(0,0)$.
(2) E is tangent to the line $\{1\} \times \mathbb{P}^1$ at the point $(1,1)$.
(3) If E meets $\mathbb{P}^1 \times \{0\}$ at $(0,0)$ and $(u,0)$ and E meets $\mathbb{P}^1 \times \{1\}$ at $(1,1)$ and $(v,1)$; then $u = v$.
(4) E passes through the point (∞, ∞).

Then E is a curve of genus 0 and we use $o = (0,0)$ as the origin of a group law on E. Let $x = (u,0)$, $p = (v,1)$ and $q = (1,1)$. We obtain the identities,

$$2[o] \simeq 2[q] \qquad\qquad [o] + [x] \simeq [p] + [q]$$
$$2[0] \simeq [x] + [p] \qquad \text{(from condition 3 above)}$$

It follows that $q = 2x$, $p = 3x$ and $4x = o$. Let $y = (\infty, infty)$. We have thus recovered the data (E, o, x, y) that we started with.

7. Forms of weight 4 and Calabi-Yau threefolds

Let E be an elliptic curve with $o \in E$ as its origin and $x \in E$ a point of order 5. Under the morphism $a : E \to |3[o]|$, the divisors $3[o]$, $[o] + [x] + [4x]$, $2[x] + [3x]$ and $2[3x] + [4x]$ are linear sections. There is a unique identification of $|3[o]|$ with $\mathbb{P}^2$ under which these sections are identified with $Z = 0$, $X = 0$, $X + Y + Z = 0$ and $Y = 0$ respectively.

Conversely, let E be a cubic curve in $\mathbb{P}^2$ which has the following properties:

(1) E passes through the points $(0:1:0)$, $(0:1:-1)$, $(1:0:-1)$ and $(0:0:1)$.
(2) The line $Z = 0$ is an inflectional tangent to E (at the point $(0:1:0)$).
(3) The line $X+Y+Z = 0$ is tangent to E at the point $(1:0:-1)$.
(4) The line $Y = 0$ is tangent to E at the point $(0:1:-1)$.

The E is a curve of genus 1. Let $o = (0:1:0)$, $x = (0:1:-1)$, $p = (0:0:1)$ and $q = (1:0:-1)$. We use o as the origin of the group law on E. We obtain the identities,

$$3[o] \simeq [o] + [x] + [p] \simeq 2[x] + [q] \simeq 2[q] + [p].$$

It follows that $q = 2x$, $p = 4x$ and $5x = o$. Thus we have obtained the data (E, o, x) that we started with.

Direct calculation shows us that the linear system of cubics in $\mathbb{P}^2$ satisfying the above conditions is the linear span of $YZ(X + Y + Z)$ and $YZ(Y + Z) - X(X + Z)^2$.

Let E be an elliptic curve with $o \in E$ as its origin and $x \in E$ a point of order 5. The morphism $a : E \to |2[o]|$ has fibres $2[o]$, $[x]+[4x]$ and $[2x]+[3x]$, which we map to 0, 1 and ∞ to identify $|2[o]|$ with $\mathbb{P}^1$. Similarly, the morphism $b : E \to |2[x]|$ has fibres $2[x]$, $[o]+[2x]$ and $[3x]+[4x]$, which we map to 0, 1 and ∞ to identify $|2[x]|$ with $\mathbb{P}^1$. Thus we obtain a morphism $a \times b : E \to \mathbb{P}^1 \times \mathbb{P}^1$, which is constructed canonically from the data (E, o, x).

Conversely, let E be a curve of type $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ which is

(1) tangent to $\mathbb{P}^1 \times \{0\}$ at the point $(1,0)$,
(2) tangent to $\{0\} \times \mathbb{P}^1$ at the point $(0,1)$,
(3) passes through the points $(\infty,1)$, (∞,∞), and $(1,\infty)$.

Then E is a curve of genus 1. Let $o=(0,1)$ and $x=(1,0)$, which are points on E. We use o as the origin of the group law on E. Let p, q, r denote the points $(\infty,1)$, (∞,∞) and $(1,\infty)$ respectively. We obtain the identities,

$$2[o] \simeq [p]+[q] \qquad 2[x] \simeq [q]+[r]$$
$$2[o] \simeq [x]+[r] \qquad 2[x] \simeq [o]+[p]$$

We solve these to show that $a=2x$, $b=3x$, $c=4x$, and $5x=o$. We have thus recovered the data (E,o,x) that we started with.

8. CM forms and C-Y varieties over suitable extensions

Let E be an elliptic curve and $n \geq 1$. Put

$$B := \{x \in E^{n+1} \mid \sum_{j=1}^{n+1} x_j = 0\},$$

which admits an action by the alternating group A_{n+1}. Consider the quotient

$$X := B/A_{n+1}.$$

The following result is proved in [PR].

Theorem *X has trivial canonical bundle, with $H^0(X,\Omega_X^p)=0$ if $0<p<n$. If $n \leq 3$, there is a smooth model $\tilde{X}$ which is Calabi-Yau.*

A submotive M of rank 4 spilts off of $H^3(\tilde{X})$ corresponding to $\mathrm{sym}^3(H^1(E))$, of Hodge type
$\{(3,0),(2,1),(1,2),(0,3)\}$. It is simple iff E is not of CM type, and in this case $\tilde{X}$ is not rigid.

Here is a sketch of proof of Theorem 2. Let Ψ be the Hecke character of an imaginary quadratic field K attached to f, so that $L(s,f) = L(s,\Psi)$. Pick an algebraic Hecke character λ of K of weight 1. Then

Ψ/λ^{k-1} is a finite order character ν. Attach a Calabi-Yau to λ^{k-1} by using the theorem above. Vary λ.

Appendix: A consequence of the Hirzebruch Riemann-Roch theorem

Theorem *The Hirzebruch Riemann-Roch theorem does not impose any restrictions on the Euler characteristic of an odd-dimensional smooth and projective variety of dimension at least 3.*

Proof Let X be any smooth projective variety. Recall that the Todd classes of a variety are the multiplicative classes defined by the generating function

$$\mathrm{td}(t) = \frac{t}{1-\exp(-t)} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \frac{t^6}{30240} + O(t^8)$$

Fix an integer m and for $i = 1, \ldots, m$, let β_i be algebraic numbers such that

$$\mathrm{td}(t) \equiv \prod_{i=1}^{m}(1+\beta_i t) \mod t^{m+1}$$

For X of dimension m, let c_i for $i = 1, \ldots, m$ be the Chern classes of its tangent bundle. Let γ_i for $i = 1, \ldots, m$ be the Chern roots so that

$$1 + c_1 t + c_2 t^2 + \cdots + c_m t^m = \prod_{i=1}^{m}(1+\gamma_i t)$$

The Todd polynomial of the variety is then given by

$$\mathrm{Todd}(t) = \prod_{i=1}^{m} td(\gamma_i t)$$

The coefficient of t^m in $\mathrm{Todd}(t)$ is the m-th Todd class Todd_m of the variety. Making use of the above expression $\mathrm{td}(t) \mod t^{m+1}$,

$$\mathrm{Todd}(t) \equiv \prod_{i,j=1}^{m}(1+\beta_j\gamma_i t) \equiv \prod_{j=1}^{m}\left(1 + c_1(\beta_j t) + c_2(\beta_j t)^2 + \cdots + c_m(\beta_j t)^m \mod t^{m+1}\right)$$

Hence, the coefficient of c_m in the m-th Todd class Todd_m is $\sum_{j=1}^{m}\beta_j$.

We can compute this as follows. Consider the function $f(t) = \log(\mathrm{td}(t))$. It has an expression modulo t^{m+1} as

$$f(t) = \sum_{j=1}^{m}\log(1+\beta_j t) = \sum_{j=1}^{m}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(\beta_j t)^k \mod t^{m+1}$$

On the other hand we have $f(-t) = f(t) - t$ so that in the expression of $f(t)$ as a power series in t, all the odd degree terms except t have coefficient 0. In particular, it follows that $\sum_{j=1}^{m} \beta_j$ is 0 whenever m is odd and $m > 1$.

To summarize, we have proved that the coefficient of the top Chern class in the Todd class of an odd-dimensional variety is 0. Since this top Chern class can be identified with the Euler characteristic we have the result that we wished to prove.

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