CHAPTER 8
DIV, GRAD, AND CURL

1. THE OPERATOR ∇ AND THE GRADIENT:

Recall that the gradient of a differentiable scalar field \( \phi \) on an open set \( D \) in \( \mathbb{R}^n \) is given by the formula:

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n} \right).
\]

(1)

It is often convenient to define formally the differential operator in vector form as:

\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right).
\]

(2)

Then we may view the gradient of \( \phi \), as the notation \( \nabla \phi \) suggests, as the result of multiplying the vector \( \nabla \) by the scalar field \( \phi \). Note that the order of multiplication matters, i.e., \( \frac{\partial \phi}{\partial x_j} \) is not \( \phi \frac{\partial}{\partial x_j} \).

Let us now review a couple of facts about the gradient. For any \( j \leq n \), \( \frac{\partial \phi}{\partial x_j} \) is identically zero on \( D \) iff \( \phi(x_1, x_2, \ldots, x_n) \) is independent of \( x_j \). Consequently,

(3) \( \nabla \phi = 0 \) on \( D \) \( \iff \) \( \phi = \) constant.

Moreover, for any scalar \( c \), we have:

(4) \( \nabla \phi \) is normal to the level set \( L_c(\phi) \).

Thus \( \nabla \phi \) gives the direction of steepest change of \( \phi \).

2. DIVERGENCE

Let \( f : D \to \mathbb{R}^n, D \subset \mathbb{R}^n \), be a differentiable vector field. (Note that both spaces are \( n \)-dimensional.) Let \( f_1, f_2, \ldots, f_n \) be the component (scalar) fields of \( f \). The divergence of \( f \) is defined to be
(5) \[ \text{div}(f) = \nabla \cdot f = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}. \]

This can be reexpressed symbolically in terms of the dot product as

(6) \[ \nabla \cdot f = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \cdot (f_1, \ldots, f_n). \]

Note that \( \text{div}(f) \) is a scalar field.

Given any \( n \times n \) matrix \( A = (a_{ij}) \), its \textbf{trace} is defined to be:

\[ \text{tr}(A) = \sum_{i=1}^{n} a_{ii}. \]

Then it is easy to see that, if \( Df \) denotes the Jacobian matrix, then

(7) \[ \nabla \cdot f = \text{tr}(Df). \]

Let \( \varphi \) be a twice differentiable scalar field. Then its \textbf{Laplacian} is defined to be

(8) \[ \nabla^2 \varphi = \nabla \cdot (\nabla \varphi). \]

It follows from (1), (5), (6) that

(9) \[ \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \cdots + \frac{\partial^2 \varphi}{\partial x_n^2}. \]

One says that \( \varphi \) is \textbf{harmonic} iff \( \nabla^2 \varphi = 0 \). Note that we can formally consider the dot product

(10) \[ \nabla \cdot \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \cdot \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \]

Then we have

(11) \[ \nabla^2 \varphi = (\nabla \cdot \nabla) \varphi. \]
Examples of harmonic functions:

(i) \( D = \mathbb{R}^2 \), \( \varphi(x, y) = e^x \cos y \).

Then \( \frac{\partial \varphi}{\partial x} = e^x \cos y \), \( \frac{\partial \varphi}{\partial y} = -e^x \sin y \), and
\( \frac{\partial^2 \varphi}{\partial x^2} = e^x \cos y \), \( \frac{\partial^2 \varphi}{\partial y^2} = -e^x \cos y \). So, \( \nabla^2 \varphi = 0 \).

(ii) \( D = \mathbb{R}^2 - \{0\} \), \( \varphi(x, y) = \log(x^2 + y^2) = 2 \log(r) \).

Then \( \frac{\partial \varphi}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial \varphi}{\partial y} = \frac{2y}{x^2 + y^2}, \frac{\partial^2 \varphi}{\partial x^2} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{-2(x^2 - y^2)}{(x^2 + y^2)^2} \), and
\( \frac{\partial^2 \varphi}{\partial y^2} = \frac{2(x^2 + y^2) - 2y(2y)}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \). So, \( \nabla^2 \varphi = 0 \).

(iii) \( D = \mathbb{R}^n - \{0\} \), \( \varphi(x_1, x_2, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\alpha/2} = r^{\alpha} \) for some fixed \( \alpha \in \mathbb{R} \).

Then \( \frac{\partial \varphi}{\partial x_i} = \alpha r^{\alpha - 1} x_i \), and
\( \frac{\partial^2 \varphi}{\partial x_i^2} = \alpha (\alpha - 2) r^{\alpha - 2} x_i \cdot x_i + \alpha r^{\alpha - 2} \cdot 1 \).

Hence \( \nabla^2 \varphi = \sum_{i=1}^{n} (\alpha (\alpha - 2) r^{\alpha - 4} x_i^2 + \alpha r^{\alpha - 2}) = \alpha (\alpha - 2 + n) r^{\alpha - 2} \).

So \( \varphi \) is harmonic for \( \alpha = 0 \) or \( \alpha = 2 - n \) (\( \alpha = -1 \) for \( n = 3 \)).

3. Cross product in \( \mathbb{R}^3 \)

The three-dimensional space is very special in that it admits a vector product, often called the cross product. Let \( i, j, k \) denote the standard basis of \( \mathbb{R}^3 \). Then, for all pairs of vectors \( v = xi + yj + zk \) and \( v' = x'i + y'j + z'k \), the cross product is defined by

\[
(12) \quad v \times v' = \det \begin{pmatrix} i & j & k \\
                        x & y & z \\
x' & y' & z' \end{pmatrix} = (yz' - y'z)i - (xz' - x'z)j + (xy' - x'y)k.
\]

Lemma 1. (a) \( v \times v' = -v' \times v \) (anti-commutativity)

(b) \( i \times j = k, j \times k = i, k \times i = j \)

(c) \( v \cdot (v \times v') = v' \cdot (v \times v') = 0 \).

Corollary: \( v \times v = 0 \).

Proof of Lemma (a) \( v' \times v \) is obtained by interchanging the second and third rows of the matrix whose determinant gives \( v \times v' \). Thus \( v' \times v = -v \times v' \).

(b) \( i \times j = \det \begin{pmatrix} i & j & k \\
                              1 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix} = k \), which is \( k \) as asserted. The other two identities are similar.

(c) \( v \cdot (v \times v') = x(yz' - y'z) - y(xz' - x'z) + z(xy' - x'y) = 0 \). Similarly for \( v' \cdot (v \times v') \).

Geometrically, \( v \times v' \) can, thanks to the Lemma, be interpreted as follows. Consider the plane \( P \) in \( \mathbb{R}^3 \) defined by \( v, v' \). Then \( v \times v' \) will
lie along the normal to this plane at the origin, and its orientation is given as follows. Imagine a corkscrew perpendicular to $P$ with its tip at the origin, such that it turns clockwise when we rotate the line $Ov$ towards $Ov'$ in the plane $P$. Then $v \times v'$ will point in the direction in which the corkscrew moves perpendicular to $P$.

Finally the length $||v \times v'||$ is equal to the area of the parallelogram spanned by $v$ and $v'$. Indeed this area is equal to the volume of the parallelepiped spanned by $v$, $v'$ and a unit vector $u = (u_x, u_y, u_z)$ orthogonal to $v$ and $v'$. We can take $u = v \times v' / ||v \times v'||$ and the (signed) volume equals

$$\det \begin{pmatrix} u_x & u_y & u_z \\ x & y & z \\ x' & y' & z' \end{pmatrix} = u_x(yz' - y'z) - u_y(xz' - x'z) + u_z(xy' - x'y)$$

$$= ||v \times v'|| \cdot (u_x^2 + u_y^2 + u_z^2) = ||v \times v'||.$$

4. Curl of vector fields in $\mathbb{R}^3$

Let $f : D \to \mathbb{R}^3$, $D \subset \mathbb{R}^3$ be a differentiable vector field. Denote by $P, Q, R$ its coordinate scalar fields, so that $f = Pi + Qj + Rk$. Then the **curl of $f$** is defined to be:

\begin{equation}
\text{curl}(f) = \nabla \times f = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}.
\end{equation}

Note that it makes sense to denote it $\nabla \times f$, as it is formally the cross product of $\nabla$ with $f$.

If the vector field $f$ represents the flow of a fluid, then the **curl** measures how the flow rotates the vectors, whence its name.

**Proposition 1.** Let $h$ (resp. $f$) be a $C^2$ scalar (resp. vector) field. Then

(a): $\nabla \times (\nabla h) = 0$.

(b): $\nabla \cdot (\nabla \times f) = 0$.

**Proof:** (a) By definition of gradient and curl,

$$\nabla \times (\nabla h) = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}$$

$$= \left( \frac{\partial^2 h}{\partial y \partial z} - \frac{\partial^2 h}{\partial z \partial y} \right) i + \left( \frac{\partial^2 h}{\partial z \partial x} - \frac{\partial^2 h}{\partial x \partial z} \right) j + \left( \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 h}{\partial y \partial x} \right) k.$$
Since $h$ is $C^2$, its second mixed partial derivatives are independent of the order in which the partial derivatives are computed. Thus, $\nabla \times (\nabla f) = 0$.

(b) By the definition of divergence and curl,
\[
\nabla \cdot (\nabla \times f) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)
\]
\[
= \left( \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} \right) + \left( -\frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 P}{\partial y \partial z} \right) + \left( \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \right).
\]
Again, since $f$ is $C^2$, $\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 P}{\partial x \partial y}$, etc., and we get the assertion.

**Warning:** There exist **twice differentiable** scalar (resp. vector) fields $h$ (resp. $f$), which are **not** $C^2$, for which (a) (resp. (b)) does **not** hold.

When the vector field $f$ represents fluid flow, it is often called **irrotational** when its curl is 0. If this flow describes the movement of water in a stream, for example, to be irrotational means that a small boat being pulled by the flow will not rotate about its axis. We will see later in this chapter the condition $\nabla \times f = 0$ occurs naturally in a purely mathematical setting as well.

**Examples:**

(i) Let $\mathcal{D} = \mathbb{R}^3 - \{0\}$ and $f(x, y, z) = \frac{y}{(x^2 + y^2)} \mathbf{i} - \frac{x}{(x^2 + y^2)} \mathbf{j}$. Show that $f$ is irrotational. Indeed, by the definition of curl,
\[
\nabla \times f = \text{det} \left( \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2 + y^2)} & \frac{x}{(x^2 + y^2)} & 0 \end{array} \right)
\]
\[
= \frac{\partial}{\partial z} \left( \frac{x}{x^2 + y^2} \right) \mathbf{i} + \frac{\partial}{\partial z} \left( \frac{y}{x^2 + y^2} \right) \mathbf{j} + \left( \frac{\partial}{\partial x} \left( \frac{-x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) \right) \mathbf{k}
\]
\[
= \left[ \frac{-(x^2 + y^2) + 2x^2}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} \right] \mathbf{k} = 0.
\]

(ii) Let $m$ be any integer $\neq 3$, $\mathcal{D} = \mathbb{R}^3 - \{0\}$, and $f(x, y, z) = \frac{1}{r^m}(x \mathbf{i} + y \mathbf{j} + z \mathbf{k})$, where $r = \sqrt{x^2 + y^2 + z^2}$. Show that $f$ is not the curl of another vector field. Indeed, suppose $f = \nabla \times g$. Then, since $f$ is $C^1$, $g$ will be $C^2$, and by the Proposition proved above, $\nabla \cdot f = \nabla \cdot (\nabla \times g)$ would be zero. But,
\[
\nabla \cdot f = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{x}{r^m}, \frac{y}{r^m}, \frac{z}{r^m} \right)
\]
\[
\begin{align*}
&= \frac{r^m - 2x^2(m^2)r^{m-2}}{r^{2m}} + \frac{r^m - 2y^2(m^2)r^{m-2}}{r^{2m}} + \frac{r^m - 2z^2(m^2)r^{m-2}}{r^{2m}} \\
&= \frac{1}{r^{2m}} (3r^m - m(x^2 + y^2 + z^2)r^{m-2}) = \frac{1}{r^m} (3 - m).
\end{align*}
\]

This is non-zero as \( m \neq 3 \). So \( f \) is not a curl.

**Warning:** It may be true that the divergence of \( f \) is zero, but \( f \) is still not a curl. In fact this happens in example (ii) above if we allow \( m = 3 \). We cannot treat this case, however, without establishing Stoke’s theorem.

5. **An Interpretation of Green’s Theorem via the Curl**

Recall that Green’s theorem for a plane region \( \Phi \) with boundary a piecewise \( C^1 \) Jordan curve \( C \) says that, given any \( C^1 \) vector field \( g = (P, Q) \) on an open set \( D \) containing \( \Phi \), we have:

\[
\int\int_\Phi \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_C P \, dx + Q \, dy.
\]

(14)

We will now interpret the term \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \). To do that, we think of the plane as sitting in \( \mathbb{R}^3 \) as \( \{z = 0\} \), and define a \( C^1 \) vector field \( f \) on \( \tilde{D} := \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D\} \) by setting \( f(x, y, z) = g(x, y) = Pi + Qj \).

Then \( \nabla \times f = \det \begin{pmatrix} i & j & k \\ P & Q & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} = \left( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) k \), because \( \frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0 \).

Thus we get:

(15)

\[
(\nabla \times f) \cdot k = \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}.
\]

And Green’s theorem becomes:

**Theorem 1.** \( \int\int_\Phi (\nabla \times f) \cdot k \, dx \, dy = \oint_C P \, dx + Q \, dy \)

6. **A Criterion for Being Conservative via the Curl**

A consequence of the reformulation above of Green’s theorem using the curl is the following:

**Proposition 1.** Let \( g : D \rightarrow \mathbb{R}^2 \), \( D \subset \mathbb{R}^2 \) open and simply connected, \( g = (P, Q) \), be a \( C^1 \) vector field. Set \( f(x, y, z) = g(x, y) \), for all \((x, y, z) \in \mathbb{R}^3 \) with \((x, y) \in D\). Suppose \( \nabla \times f = 0 \). Then \( g \) is conservative on \( D \).
Proof: Since $\nabla \times f = 0$, the reformulation in section 5 of Green’s theorem implies that $\oint_C P\,dx + Q\,dy = 0$ for all Jordan curves $C$ contained in $D$. QED

Example: $D = \mathbb{R}^2 - \{(x, 0) \in \mathbb{R}^2 \mid x \leq 0\}$, $g(x, y) = \frac{y}{x^2+y^2}i - \frac{x}{x^2+y^2}j$. Determine if $g$ is conservative on $D$:

Again, define $f(x, y, z)$ to be $g(x, y)$ for all $(x, y, z)$ in $\mathbb{R}^3$ such that $(x, y) \in D$. Since $g$ is evidently $C^1$, $f$ will be $C^1$ as well. By the Proposition above, it will suffice to check if $f$ is irrotational, i.e., $\nabla \times f = 0$, on $D \times \mathbb{R}$. This was already shown in Example (i) of section 4 of this chapter. So $g$ is conservative.