Chapter 3

Differentiation in higher dimensions

3.1 The Total Derivative

Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a 1-variable function, and $a \in \mathbb{R}$, we say that $f$ is differentiable at $x = a$ if and only if the ratio $\frac{f(a+h)-f(a)}{h}$ tends to a finite limit, denoted $f'(a)$, as $h$ tends to 0.

There are two possible ways to generalize this for vector fields

$$f : \mathcal{D} \to \mathbb{R}^m, \mathcal{D} \subseteq \mathbb{R}^n,$$

for points $a$ in the interior $\mathcal{D}^0$ of $\mathcal{D}$. (The interior of a set $X$ is defined to be the subset $X^0$ obtained by removing all the boundary points. Since every point of $X^0$ is an interior point, it is open.) The reader seeing this material for the first time will be well advised to stick to vector fields $f$ with domain all of $\mathbb{R}^n$ in the beginning. Even in the one dimensional case, if a function is defined on a closed interval $[a, b]$, say, then one can properly speak of differentiability only at points in the open interval $(a, b)$.

The first thing one might do is to fix a vector $v$ in $\mathbb{R}^n$ and say that $f$ is differentiable along $v$ iff the following limit makes sense:

$$\lim_{h \to 0} \frac{1}{h} (f(a + hv) - f(a)).$$

When it does, we write $f'(a; v)$ for the limit. Note that this definition makes sense because $a$ is an interior point. Indeed, under this hypothesis, $\mathcal{D}$ contains a basic open set $U$ containing $a$, and so $a + hv$ will, for small enough $h$, fall into $U$, allowing us to speak of $f(a + hv)$. This
derivative behaves exactly like the one variable derivative and has analogous properties. For example, we have the following

**Mean Value Theorem**  Assume \( f'(a + tv; v) \) exists for all \( 0 \leq t \leq 1 \). Then \( \exists t_0 \in [0, 1] \) such that \( f'(a + t_0v; v) = f(a + v) - f(a) \).

**Proof.** Put \( \phi(t) = f(a + tv) \). By hypothesis, \( \phi \) is differentiable at every \( t \) in \([0, 1]\), and \( \phi'(t) = f'(a + tv; v) \). By the one variable mean value theorem, there exists a \( t_0 \) such that \( \phi'(t_0) = \phi(1) - \phi(0) \), which equals \( f(a + v) - f(a) \). Done.

When \( v \) is a unit vector, \( f'(a; v) \) is called the **directional derivative** of \( f \) at \( a \) in the direction of \( v \).

The disadvantage of this construction is that it forces us to study the change of \( f \) in one direction at a time. So we revisit the one-dimensional definition and note that the condition for differentiability there is equivalent to requiring that there exists a constant \( c = f'(a) \), such that \( \lim_{h \to 0} \frac{f(a + h) - f(a) - ch}{h} = 0 \). If we put \( L(h) = f'(a)h \), then \( L : \mathbb{R} \to \mathbb{R} \) is clearly a linear map. We generalize this idea in higher dimensions as follows:

**Definition.** Let \( f : D \to \mathbb{R}^m \) \((D \subseteq \mathbb{R}^n)\) be a vector field and \( a \) an interior point of \( D \). Then \( f \) is differentiable at \( x = a \) if and only if there exists a linear map \( L : \mathbb{R}^n \to \mathbb{R}^m \) such that

\[
\lim_{u \to 0} \frac{||f(a + u) - f(a) - L(u)||}{||u||} = 0.
\]

Note that the norm \( || \cdot || \) denotes the length of vectors in \( \mathbb{R}^m \) in the numerator and in \( \mathbb{R}^n \) in the denominator; this should not lead to any confusion, however.

**Lemma 1**  Such an \( L \), if it exists, is unique.

**Proof.** Suppose we have \( L, M : \mathbb{R}^n \to \mathbb{R}^m \) satisfying (*) at \( x = a \). Then

\[
\lim_{u \to 0} \frac{||L(u) - M(u)||}{||u||} = \lim_{u \to 0} \frac{||L(u) + f(a) - f(a + u) + (f(a + u) - f(a) - M(u))||}{||u||} \\
\leq \lim_{u \to 0} \frac{||L(u) + f(a) - f(a + u)||}{||u||} \\
+ \lim_{u \to 0} \frac{||f(a + u) - f(a) - M(u)||}{||u||} = 0.
\]
Pick any non-zero $v \in \mathbb{R}^n$, and set $u = tv$, with $t \in \mathbb{R}$. Then, the linearity of $L, M$ implies that $L(tv) = tL(v)$ and $M(tv) = tM(v)$. Consequently, we have

$$
\lim_{t \to 0} \frac{||L(tv) - M(tv)||}{||tv||} = 0
$$

$$
= \lim_{t \to 0} \frac{|t||L(v) - M(v)||}{|t||v||}
$$

$$
= \frac{1}{||v||}||L(v) - M(v)||.
$$

Then $L(v) - M(v)$ must be zero.

**Definition.** If the limit condition $(*)$ holds for a linear map $L$, we call $L$ the **total derivative** of $f$ at $a$, and denote it by $T_a f$.

It is mind boggling at first to think of the derivative as a linear map. A natural question which arises immediately is to know what the value of $T_a f$ is at any vector $v$ in $\mathbb{R}^n$. We will show in section 4.3 that this value is precisely $f'(a; v)$, thus linking the two generalizations of the one-dimensional derivative.

Sometimes one can guess what the answer should be, and if $(*)$ holds for this choice, then it must be the derivative by uniqueness. Here are two examples which illustrate this.

1. Let $f$ be a **constant vector field**, i.e., there exists a vector $w \in \mathbb{R}^m$ such that $f(x) = w$, for all $x$ in the domain $D$. Then we claim that $f$ is differentiable at any $a \in D^0$ with derivative zero. Indeed, if we put $L(u) = 0$, for any $u \in \mathbb{R}^n$, then $(*)$ is satisfied, because $f(a + u) - f(a) = w - w = 0$.

2. Let $f$ be a **linear map**. Then we claim that $f$ is differentiable everywhere with $T_a f = f$. Indeed, if we put $L(u) = f(u)$, then by the linearity of $f$, $f(a + u) - f(a) - L(u)$ will be zero for any $u \in \mathbb{R}^n$, so that $(*)$ holds trivially.

Before we leave this section, it will be useful to take note of the following:

**Lemma 2** Let $f_1, \ldots, f_m$ be the component (scalar) fields of $f$. Then $f$ is differentiable at $a$ iff each $f_i$ is differentiable at $a$.

An easy consequence of this lemma is that, when $n = 1$, $f$ is differentiable at $a$ iff the following familiar looking limit exists in $\mathbb{R}^m$:

$$
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},
$$
allowing us to suggestively write $f'(a)$ instead of $T_a f$. Clearly, $f'(a)$ is given by the vector $(f'_1(a), \ldots, f'_m(a))$, so that $(T_a f)(h) = f'(a)h$, for any $h \in \mathbb{R}$.

**Proof.** Let $f$ be differentiable at $a$. For each $v \in \mathbb{R}^n$, write $L_i(v)$ for the $i$-th component of $(T_a f)(v)$. Then $L_i$ is clearly linear. Since $f_i(a + u) - f_i(u) - L_i(u)$ is the $i$-th component of $f(a + u) - f(a) - L(u)$, the norm of the former is less than or equal to that of the latter. This shows that (*) holds with $f$ replaced by $f_i$ and $L$ replaced by $L_i$. So $f_i$ is differentiable for any $i$. Conversely, suppose each $f_i$ differentiable. Put $L(v) = ((T_a f_1)(v), \ldots, (T_a f_m)(v))$. Then $L$ is a linear map, and by the triangle inequality,

$$||f(a + u) - f(a) - L(u)|| \leq \sum_{i=1}^{m} |f_i(a + u) - f_i(a) - (T_a f_i)(u)|.$$ 

It follows easily that (*) exists and so $f$ is differentiable at $a$.

### 3.2 Partial Derivatives

Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{R}^n$. The directional derivatives along the unit vectors $e_j$ are of special importance.

**Definition.** Let $j \leq n$. The $j$th partial derivative of $f$ at $x = a$ is $f'(a; e_j)$, denoted by $\frac{\partial f}{\partial x_j}(a)$ or $D_j f(a)$.

Just as in the case of the total derivative, it can be shown that $\frac{\partial f}{\partial x_j}(a)$ exists iff $\frac{\partial f_i}{\partial x_j}(a)$ exists for each coordinate field $f_i$.

**Example:** Define $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$f(x, y, z) = (e^{x \sin(y)}, z \cos(y)).$$

All the partial derivatives exist at any $a = (x_0, y_0, z_0)$. We will show this for $\frac{\partial f}{\partial y}$ and leave it to the reader to check the remaining cases. Note that

$$\frac{1}{h} f(a + he_2) - f(a) = \left(\frac{e^{x_0 \sin(y_0 + h)} - e^{x_0 \sin(y_0)}}{h}, z_0 \frac{\cos(y_0 + h) - \cos(y_0)}{h}\right).$$

We have to understand the limit as $h$ goes to 0. Then the methods of one variable calculus show that the right hand side tends to the finite limit $(x_0 \cos(y_0)e^{x_0 \sin(y_0)}, -z_0 \sin(y_0))$, which
is \( \frac{\partial f}{\partial y}(a) \). In effect, the partial derivative with respect to \( y \) is calculated like a one variable derivative, keeping \( x \) and \( z \) fixed. Let us note without proof that \( \frac{\partial f}{\partial x}(a) = (\cos(y_0) e^{x_0 \sin(y_0)}, 0) \) and \( \frac{\partial f}{\partial z}(a) = (0, \cos y_0) \).

It is easy to see from the definition that \( f'(a ; tv) \) equals \( tf(a; v) \), for any \( t \in \mathbb{R} \). We also have the following

**Lemma 3** Suppose the derivatives of \( f \) along any \( v \in \mathbb{R}^n \) exist near \( a \) and are continuous at \( a \). Then

\[
 f'(a; v + v') = f'(a; v) + f'(a; v'),
\]

for all \( v, v' \) in \( \mathbb{R}^n \). In particular, the directional derivatives of \( f \) are all determined by the \( n \) partial derivatives.

**Proof.** If \( \phi, \psi \) are functions of \( h \in \mathbb{R} \), let us write

\[
\phi(h) \equiv \psi(h) \iff \lim_{h \to 0} \frac{\phi(h) - \psi(h)}{h} = 0.
\]

Check that \( \equiv \) is an equivalence relation. Then by definition, we have, for all \( a \in D^0 \) and \( u \) in \( \mathbb{R}^n \),

\[
 f(a + hu) \equiv f(a) + hf'(a; u).
\]

Then \( f(a + h(v + v')) \) is equivalent to \( f(a) + hf'(a; v + v') \) on the one hand, and to

\[
 f(a + hv) + hf'(a + hv; v') \equiv f(a) + h(f'(a; v) + f'(a + hv; v')),
\]

on the other. Moreover, the continuity hypothesis shows that \( f'(a + hv; v') \) tends to \( f'(a; v') \) as \( h \) goes to 0. Consequently, we get the equivalence of \( f'(a; v + v') \) with \( f'(a; v) + f'(a; v') \).

Since they are independent of \( h \), they must in fact be equal.

Finally, since \( \{e_j | j \leq n\} \) is a basis of \( \mathbb{R}^n \), we can write any \( v \) as \( \sum_j \alpha_j e_j \), and by what we have just shown, \( f'(a : v) \) is determined as \( \sum_j \alpha_j \frac{\partial f}{\partial x_j}(a) \).

In the next section we will show that the conclusion of this lemma remains valid without the continuity hypothesis if we assume instead that \( f \) has a total derivative at \( a \).
The gradient of a scalar field \( g \) at an interior point \( a \) of its domain in \( \mathbb{R}^n \) is defined to be the following vector in \( \mathbb{R}^n \):

\[
\nabla g(a) = \text{grad} g(a) = \left( \frac{\partial g}{\partial x_1}(a), \ldots, \frac{\partial g}{\partial x_n}(a) \right).
\]

Given a vector field \( f \) as above, we can then put together the gradients of its component fields \( f_i, \ 1 \leq i \leq m \), and form the following important matrix, called the Jacobian matrix at \( a \):

\[
Df(a) = \left( \frac{\partial f_i}{\partial x_j}(a) \right)_{1 \leq i \leq m, 1 \leq j \leq n} \in M_{mn}.
\]

The \( i \)-th row is given by \( \nabla f_i(a) \), while the \( j \)-th column is given by \( \frac{\partial f}{\partial x_j}(a) \).

### 3.3 The main theorem

In this section we collect the main properties of the total and partial derivatives.

**Theorem 1** Let \( f : D \to \mathbb{R}^m \) be a vector field, and \( a \) an interior point of its domain \( D \subseteq \mathbb{R}^n \).

(a) If \( f \) is differentiable at \( a \), then for any vector \( v \) in \( \mathbb{R}^n \),

\[
(Ta f)(v) = f'(a, v).
\]

In particular, since \( Ta f \) is linear, we have

\[
f'(a; \alpha v + \beta v') = \alpha f'(a; v) + \beta f'(a; v'),
\]

for all \( v, v' \) in \( \mathbb{R}^n \) and \( \alpha, \beta \) in \( \mathbb{R} \).

(b) Again assume that \( f \) is differentiable. Then the matrix of the linear map \( Ta f \) relative to the standard bases of \( \mathbb{R}^n, \mathbb{R}^m \) is simply the Jacobian matrix of \( f \) at \( a \).

(c) \( f \) differentiable at \( a \) \( \Rightarrow \) \( f \) continuous at \( a \).

(d) Suppose all the partial derivatives of \( f \) exist near \( a \) and are continuous at \( a \). Then \( Ta f \) exists.
(e) **(chain rule)** Consider

\[ \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^h. \]

Suppose \( f \) is differentiable at \( a \) and \( g \) is differentiable at \( b = f(a) \). Then the composite function \( h = g \circ f \) is differentiable at \( a \) and moreover,

\[ T_ah = T_bg \circ T_af. \]

In terms of the Jacobian matrices, this reads as

\[ Dh(a) = Dg(b)Df(a) \in M_{kn}. \]

(f) \((m = 1)\) Let \( f, g \) be scalar fields, differentiable at \( a \). Then

(i) \( T_a(f + g) = T_af + T_ag \) \quad (additivity)

(ii) \( T_a(fg) = f(a)T_ag + g(a)T_af \) \quad (product rule)

(iii) \( T_a\left(\frac{f}{g}\right) = \frac{g(a)T_af - f(a)T_ag}{g(a)^2} \) if \( g(a) \neq 0 \) \quad (quotient rule)

The following corollary is an immediate consequence of the theorem, which we will make use of in the next chapter on normal vectors and extrema.

**Corollary 1** Let \( g \) be a scalar field, differentiable at an interior point \( b \) of its domain \( D \) in \( \mathbb{R}^n \), and let \( v \) be any vector in \( \mathbb{R}^n \). Then we have

\[ \nabla g(b) \cdot v = f'(b;v). \]

Furthermore, let \( \phi \) be a function from a subset of \( \mathbb{R} \) into \( D \subseteq \mathbb{R}^n \), differentiable at an interior point \( a \) mapping to \( b \). Put \( h = g \circ \phi \). Then \( h \) is differentiable at \( a \) with

\[ h'(a) = \nabla g(b) \cdot \phi'(a). \]

Here is a simple observation before we begin the proof. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a vector field such that \( f_1(x,y) = \phi(x), f_2(x,y) = \psi(y) \), with \( \phi, \psi \) differentiable everywhere. Then, clearly, the Jacobian matrix \( Df(x,y) \) is the **diagonal matrix** \( \begin{pmatrix} \phi'(x) & 0 \\ 0 & \psi'(y) \end{pmatrix} \). Conversely,
By definition, \( f \) is the limit \( \lim_{n \to 0} \frac{f_1(x, y) - f_1(a, y) - (f_1(a, y) - f_1(a, y))}{n} = 0 \). Then \( \frac{\partial f_1}{\partial x} = \mu(x) \), \( \frac{\partial f_1}{\partial y} = 0 = \frac{\partial f_2}{\partial x} \), \( \frac{\partial f_2}{\partial y} = \nu(y) \) \( \Rightarrow f_1(x, y) = \int \mu(x) \, dx \); \( f_2(x, y) = \int \nu(y) \, dy \). So \( f_1 \) is independent of \( y \) and \( f_2 \) is independent of \( x \).

**Proof of main theorem.** (a) It suffices to show that \( (T_a f_i)(v) = f_i(a; v) \) for each \( i \leq n \). By definition,

\[
\lim_{u \to 0} \frac{||f_i(a + u) - f_i(a) - (T_a f_i)(u)||}{||u||} = 0
\]

This means that we can write for \( u = hv, h \in \mathbb{R} \),

\[
\lim_{h \to 0} \frac{f_i(a + hv) - f_i(a) - h(T_a f_i)(v)}{||h|| ||v||} = 0.
\]

In other words, the limit \( \lim_{h \to 0} \frac{f_i(a + hv) - f_i(a)}{h} \) exists and equals \( (T_a f_i)(v) \). Done.

(b) By part (a), each partial derivative exists at \( a \) (since \( f \) is assumed to be differentiable at \( a \)). The matrix of the linear map \( T_a f \) is determined by the effect on the standard basis vectors. Let \( \{e'_i|1 \leq i \leq m\} \) denote the standard basis in \( \mathbb{R}^m \). Then we have, by definition,

\[
(T_a f)(e_j) = \sum_{i=1}^{m} (T_a f_i)(e_j)e'_i = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(a)e'_i.
\]

The matrix obtained is easily seen to be \( Df(a) \).

(c) Suppose \( f \) is differentiable at \( a \). This certainly implies that the limit of the function \( f(a + u) - f(a) - (T_a f)(u) \), as \( u \) tends to \( 0 \in \mathbb{R}^n \), is \( 0 \in \mathbb{R}^m \) (from the very definition of \( T_a f \), \( ||f(a + u) - f(a) - (T_a f)(u)|| \) tends to zero „faster” than \( ||u|| \), in particular it tends to zero). Since \( T_a f \) is linear, \( T_a f \) is continuous (everywhere), so that \( \lim_{u \to 0}(T_a f)(u) = 0 \). Hence \( \lim_{u \to 0} f(a + u) = f(a) \) which means that \( f \) is continuous at \( a \).

(d) By hypothesis, all the partial derivatives exist near \( a = (a_1, \ldots, a_n) \) and are continuous there. It suffices to show that each \( f_i \) is differentiable at \( a \). So we have only to show that (*) holds with \( f \) replaced by \( f_i \) and \( L(u) = f_i^*(a; u) \). Write \( u = (h_1, \ldots, h_n) \). By Lemma 3, we know that \( f_i^*(a; -) \) is linear. So

\[
L(u) = \sum_{j=1}^{n} h_j \frac{\partial f_i}{\partial x_j}(a),
\]

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and we can write
\[ f_i(a + u) - f_i(a) = \sum_{j=1}^{n} (\phi_j(a_j + h_j) - \phi_j(a_j)), \]
where each \( \phi_j \) is a one variable function defined by
\[ \phi_j(t) = f_i(a_1 + h_1, \ldots, a_{j-1} + h_{j-1}, t, a_{j+1}, \ldots, a_n). \]

By the mean value theorem,
\[ \phi_j(a_j + h_j) - \phi_j(a_j) = \phi_j'(t_j) = \frac{\partial f_i}{\partial x_j}(y(j)), \]
for some \( t_j \in [a_j, a_j + h_j] \), with
\[ y(j) = (a_1 + h_1, \ldots, a_{j-1} + h_{j-1}, t_j, a_{j+1}, \ldots, a_n). \]

Putting these together, we see that it suffices to show that the following limit is zero:
\[ \lim_{u \to 0} \frac{1}{||u||} \sum_{j=1}^{n} h_j \left( \frac{\partial f_i}{\partial x_j}(a) - \frac{\partial f_i}{\partial x_j}(y(j)) \right). \]

Clearly, \( |h_j| \leq ||u|| \), for each \( j \). So it follows, by the triangle inequality, that this limit is bounded above by the sum over \( j \) of \( \lim_{h_j \to 0} |(\frac{\partial f_i}{\partial x_j}(a) - \frac{\partial f_i}{\partial x_j}(y(j))| \), which is zero by the continuity of the partial derivatives at \( a \). Here we are using the fact that each \( y(j) \) approaches \( a \) as \( h_j \) goes to 0. Done.

(e) First we need the following simple

**Lemma 4** Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear map. Then, \( \exists c > 0 \) such that \( ||Tv|| \leq c||v|| \) for any \( v \in \mathbb{R}^n \).

**Proof of Lemma.** Let \( A \) be the matrix of \( T \) relative to the standard bases. Put \( C = \max_j{||T(e_j)||} \). If \( v = \sum_{j=1}^{n} \alpha_j e_j \), then
\[ ||T(v)|| = || \sum_{j} \alpha_j T(e_j)|| \leq C \sum_{j=1}^{n} |\alpha_j| \cdot 1 \]
\[ \leq C\left( \sum_{j=1}^{n} |\alpha_j|^2 \right)^{1/2} \left( \sum_{j=1}^{n} 1 \right)^{1/2} \leq C\sqrt{n}||v||, \]
by the Cauchy–Schwarz inequality. We are done by setting $c = C\sqrt{n}$.

**Proof of (e) (contd.).** Write $L = T_\alpha f$, $M = T_\beta g$, $N = M \circ L$. To show: $T_\alpha h = N$.

Define $F(x) = f(x) - f(a) - L(x - a)$, $G(y) = g(y) - g(b) - M(y - b)$ and $H(x) = h(x) - h(a) - N(x - a)$. Then we have

$$\lim_{x \to a} \frac{||F(x)||}{||x - a||} = 0 = \lim_{y \to b} \frac{||G(y)||}{||y - b||}.$$ 

So we need to show:

$$\lim_{x \to a} \frac{||H(x)||}{||x - a||} = 0.$$

But

$$H(x) = g(f(x)) - g(b) - M(L(x - a))$$

Since $L(x - a) = f(x) - f(a) - F(x)$, we get

$$H(x) = [g(f(x)) - g(b) - M(f(x) - f(a))] + M(F(x)) = G(f(x)) + M(F(x)).$$

Therefore it suffices to prove:

(i) $\lim_{x \to a} \frac{||G(f(x))||}{||x - a||} = 0$ and

(ii) $\lim_{x \to a} \frac{||M(F(x))||}{||x - a||} = 0$.

By Lemma 4, we have $||M(F(x))|| \leq c||F(x)||$, for some $c > 0$. Then $\frac{||M(F(x))||}{||x - a||} \leq c \lim_{x \to a} \frac{||F(x)||}{||x - a||} = 0$, yielding (ii).

On the other hand, we know $\lim_{y \to b} \frac{||G(y)||}{||y - b||} = 0$. So we can find, for every $\epsilon > 0$, a $\delta > 0$ such that $||G(f(x))|| < \epsilon ||f(x) - b||$ if $||f(x) - b|| < \delta$. But since $f$ is continuous, $||f(x) - b|| < \delta$ whenever $||x - a|| < \delta_1$, for a small enough $\delta_1 > 0$. Hence

$$||G(f(x))|| < \epsilon ||f(x) - b|| = \epsilon ||F(x) + L(x - a)|| \leq \epsilon ||F(x)|| + \epsilon ||L(x - a)||,$$

by the triangle inequality. Since $\lim_{x \to a} \frac{||F(x)||}{||x - a||}$ is zero, we get

$$\lim_{x \to a} \frac{||G(f(x))||}{||x - a||} \leq \epsilon \lim_{x \to a} \frac{||L(x - a)||}{||x - a||}.$$
Applying Lemma 4 again, we get $||L(x - a)|| \leq c'||x - a||$, for some $c' > 0$. Now (i) follows easily.

(f) (i) We can think of $f + g$ as the composite $h = s(f, g)$ where $(f, g)(x) = (f(x), g(x))$ and $s(u, v) = u + v$ (“sum”). Set $b = (f(a), g(a))$. Applying (e), we get

$$T_a(f + g) = T_b(s) \circ T_a(f, g) = T_a(f) + T_b(g).$$

Done. The proofs of (ii) and (iii) are similar and will be left to the reader.

QED.

Remark. It is important to take note of the fact that a vector field $f$ may be differentiable at $a$ without the partial derivatives being continuous. We have a counterexample already when $n = m = 1$ as seen by taking

$$f(x) = x^2 \sin \left( \frac{1}{x} \right) \text{ if } x \neq 0,$$

and $f(0) = 0$. This is differentiable everywhere. The only question is at $x = 0$, where the relevant limit $\lim_{h \to 0} \frac{f(h)}{h}$ is clearly zero, so that $f'(0) = 0$. But for $x \neq 0$, we have by the product rule,

$$f'(x) = 2x \sin \left( \frac{1}{x} \right) - \cos \left( \frac{1}{x} \right),$$

which does not tend to $f'(0) = 0$ as $x$ goes to 0. So $f'$ is not continuous at 0.

### 3.4 Mixed partial derivatives

Let $f$ be a scalar field, and $a$ an interior point in its domain $\mathcal{D} \subseteq \mathbb{R}^n$. For $j, k \leq n$, we may consider the second partial derivative

$$\frac{\partial^2 f}{\partial x_j \partial x_k}(a) = \frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_k} \right)(a),$$

when it exists. It is called the mixed partial derivative when $j \neq k$, in which case it is of interest to know whether we have the equality

$$\frac{\partial^2 f}{\partial x_j \partial x_k}(a) = \frac{\partial^2 f}{\partial x_k \partial x_j}(a).$$

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Proposition 1 Suppose $\frac{\partial^2 f}{\partial x_j \partial x_k}$ and $\frac{\partial^2 f}{\partial x_k \partial x_j}$ both exist near $a$ and are continuous there. Then the equality (3.4.1) holds.

The proof is similar to the proof of part (d) of Theorem 1.