SET THEORY AND UNIQUENESS FOR TRIGONOMETRIC SERIES

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Dedicated to the memory of my friend and colleague Stelios Pichorides

Problems concerning the uniqueness of an expansion of a function in a trigonometric series have a long and fascinating history, starting back in the 19th Century with the work of Riemann, Heine and Cantor. The origins of set theory are closely connected with this subject, as it was Cantor's research into the nature of exceptional sets for such uniqueness problems that led him to the creation of set theory. And the earliest application of one of Cantor's fundamental concepts, that of ordinal numbers and transfinite induction, can be glimpsed in his last work on this subject.

The purpose of this paper is to give a basic introduction to the application of set theoretic methods to problems concerning uniqueness for trigonometric series. It is written in the style of informal lecture notes for a course or seminar on this subject and, in particular, contains several exercises. The treatment is as elementary as possible and only assumes some familiarity with the most basic results of general topology, measure theory, functional analysis, and descriptive set theory. Standard references to facts that are used without proof are given in the appropriate places.

The notes are divided into three parts. The first deals with ordinal numbers and transfinite induction, and gives an exposition of Cantor's work. The second gives an application of Baire category methods, one of the basic set theoretic tools in the arsenal of an analyst. The final part deals with the role of descriptive methods in the study of sets of uniqueness. It closes with an introduction to the modern use of ordinal numbers in descriptive set theory and its applications, and brings us back full circle to the concepts that arose in the beginning of this subject.

There is of course much more to this area than what is discussed in these introductory notes. Some references for further study include: Kechris-Louveau [1989, 1992], Cooke [1993], Kahane-Salem [1994], Lyons [1995]. The material in these notes is mostly drawn from these sources.

Stelios Pichorides always had a strong interest in the problems of uniqueness for trigonometric series, and, in particular, in the problem of characterizing the sets of uniqueness. I remember distinctly a discussion we had, in the early eighties, during a long ride in the Los Angeles freeways, in which he wondered whether lack of further progress on this

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deep problem perhaps had some logical or foundational explanation. (It should be pointed out that mathematical logic was Stelios' first love, and he originally went to the University of Chicago to study this field.) His comments are what got me interested in this subject and we have had extensive discussions about this for many years afterwards. Certainly the results proved in the 1980's (some of which are discussed in Part III below) concerning the Characterization Problem seem to confirm his intuition.

Problems concerning sets of uniqueness have fascinated mathematicians for over 100 years now, in part because of the intrinsic nature of the subject and in part because of its intriguing interactions with other areas of classical analysis, measure theory, functional analysis, number theory, and set theory. Once someone asked Paul Erdös, after he gave a talk about one of his favorite number theory problems, somewhat skeptically, why he was so interested in this problem. Erdös replied that if this problem was good enough for Dirichlet and Gauss, it was good enough for him. To paraphrase Erdös, if the problems of uniqueness for trigonometric series were good enough for Riemann, Cantor, Luzin, Menshov, Bari, Salem, Zygmund, and Pichorides, they are certainly good enough for me.

PART I. ORDINAL NUMBERS AND TRANSFINITE INDUCTION.

§1. The problem of uniqueness for trigonometric series.

A *trigonometric series* S is an infinite series of the form

$$S \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $a_n \in \mathbb{C}, x \in \mathbb{R}$. We view this as a formal expression without any claims about its convergence at a given point x.

The Nth *partial sum* of this series is the *trigonometric polynomial*

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx).$$

If, for some given x, $S_N(x) \to s \in \mathbb{C}$, we write

$$s = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and call s the sum of the series at x.

A function $f : \mathbb{R} \to \mathbb{C}$ *admits a trigonometric expansion* if there is a series S as above so that for every $x \in \mathbb{R}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

It is clear that any such function is periodic with period 2π.

It is a very difficult problem to characterize the functions f which admit a trigonometric expansion, but it is a classical result that any "nice" enough 2π-periodic function f, for example a continuously differentiable one, admits a trigonometric expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

there the coefficients a_n, b_n can be, in fact, computed by the well-known *Fourier formulas*

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt.$$

The following question now arises naturally: *Is such an expansion unique*?

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{a'_0}{2} + \sum_{n=1}^{\infty}(a'_n \cos nx + b'_n \sin nx)$, then, by subtracting, we would have a series $\frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos nx + d_n \sin nx)$ with

$$0 = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos nx + d_n \sin nx)$$

for every x. So the problem is equivalent to the following:

Uniqueness Problem. *If* $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *is a trigonometric series and*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 0$$

for all $x \in \mathbb{R}$, *it is true that* $a_n = b_n = 0$ *for all* n?

This is the problem (which arose through the work of Riemann and Heine) that Heine proposed, in 1869, to the 24 year old Cantor, who had just accepted a position at the university in Halle, where Heine was a senior colleague.

In the next few sections we will give Cantor's solution to the uniqueness problem and see how his search for extensions, allowing exceptional points, led him to the creation of set theory, including the concepts of ordinal numbers and the method of transfinite induction. We will also see how this method can be used to prove the first such major extension.

Before we proceed, it would be convenient to also introduce an alternative form for trigonometric series.

Every series

$$S \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

can be also written as

$$S \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where, letting $b_0 = 0$,

$$c_n = \frac{a_n - ib_n}{2}, c_{-n} = \frac{a_n + ib_n}{2} \quad (n \in \mathbb{N}).$$

(Thus $a_n = c_n + c_{-n}, b_n = i(c_n - c_{-n})$.) In this notation, the partial sums $S_N(x)$ are given by

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx};$$

and if they converge as $N \to \infty$ with limit s, we write

$$s = \sum_{n=-\infty}^{+\infty} c_n e^{inx}.$$

The standard examples of trigonometric series are the *Fourier series* of integrable functions. Given a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ we say that it is *integrable* if it is Lebesgue measurable and

$$\frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt < \infty.$$

In this case we define its *Fourier coefficients* $\hat{f}(n), n \in \mathbb{Z}$, by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt.$$

We call the trigonometric series

$$S(f) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

the *Fourier series* of f, and write

$$S_N(f, x) = \sum_{n=-N}^{+N} \hat{f}(n) e^{inx}$$

for its partial sums.

Remark. There are trigonometric series, like $\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$, which converge everywhere but are not Fourier series.

§2. The Riemann Theory.

Riemann was the first mathematician to seriously study general trigonometric series (as opposed to Fourier series), in his Habilitationsschrift (1854). We will prove here two of his main results, that were used by Cantor. These results are beautiful applications of elementary calculus.

Let $S \sim \sum c_n e^{inx}$ be an arbitrary trigonometric series with *bounded coefficients,*i.e., $|c_n| \leq M$ for some M and all $n \in \mathbb{Z}$. Then Riemann had the brilliant idea to consider the function obtained by formally integrating $\sum c_n e^{inx}$ twice. This function, called the *Riemann function* F_S of S, is thus defined by

$$F_S(x) = \frac{c_0 x^2}{2} - \sum_{n=-\infty}^{+\infty}{}' \frac{1}{n^2} c_n e^{inx}, x \in \mathbb{R},$$

where $\sum'$ means $n = 0$ is excluded.

Clearly, as c_n is bounded, F_S is a continuous function on $\mathbb{R}$ (but is not periodic), since the above series converges absolutely and uniformly, as $|\frac{1}{n^2}c_n e^{inx}| \leq \frac{M}{n^2}$.

Now, intuitively, one would hope that $F''_S(x)$ should be the same as $\sum c_n e^{inx}$, if this sum exists. This may not be quite true, but something close enough to it is.

Given a function $F : \mathbb{R} \to \mathbb{C}$, let

$$\Delta^2 F(x, h) = F(x + h) + F(x - h) - 2F(x)$$

and define the *second symmetric derivative* or *second Schwartz derivative* of F at x by

$$D^2 F(x) = \lim_{h\to 0} \frac{\Delta^2 F(x, h)}{h^2},$$

provided this limit exists.

2.1. Exercise. If $F''(x)$ exists, then so does $D^2F(x)$ and they are equal, but the converse fails.

2.2. Riemann's First Lemma. *Let $S \sim \sum c_n e^{inx}$ be a trigonometric series with bounded coefficients. If $s = \sum c_n e^{inx}$ exists, then $D^2F_S(x)$ exists and $D^2F_S(x) = s$.*

Proof. We have by calculating

$$\frac{\Delta^2 F_S(x, 2h)}{4h^2} = \sum_{n=-\infty}^{+\infty} \left[\frac{\sin nh}{nh}\right]^2 c_n e^{inx}$$

(for $n = 0$, we let $\frac{\sin nh}{nh} = 1$). So it is enough to prove the following:

2.3. Lemma. $\sum\limits_{n=0}^{\infty} a_n = a \Rightarrow \lim\limits_{h\to 0} \left[\sum\limits_{n=0}^{\infty} \left[\frac{\sin nh}{nh}\right]^2 a_n \right] = a.$

Proof. Put $A_N = \sum\limits_{n=0}^{N} a_n$. Then

$$\sum_{n=0}^{\infty} \left(\frac{\sin nh}{nh}\right)^2 a_n = \sum_{n=0}^{\infty} \left[\left(\frac{\sin nh}{nh}\right)^2 - \left(\frac{\sin(n+1)h}{(n+1)h}\right)^2 \right] A_n.$$

Let $h_k \to 0, h_k > 0$ and put

$$s_{kn} = \left(\frac{\sin nh_k}{nh_k}\right)^2 - \left(\frac{\sin(n+1)h_k}{(n+1)h_k}\right)^2.$$

Then we have to show that

$$A_n \xrightarrow{n} a \Rightarrow \sum_{n=0}^{\infty} A_n s_{kn} \xrightarrow{k} a.$$

We view the infinite matrix (s_{kn}) as a *summability method,* i.e., a way of transforming a sequence (x_n) into the sequence

$$(y_k) = (s_{kn}) \cdot (x_n),$$

i.e., $y_k = \sum\limits_{n=0}^{\infty} s_{kn} x_n$.

Example. If $s_{kn} = \frac{1}{k+1}$ for $n \le k$, $s_{kn} = 0$ for $n > k$, then $y_k = \frac{x_0 + \cdots + x_k}{k+1}$.

A summability method is called *regular* if $x_n \xrightarrow{n} x \Rightarrow y_k \xrightarrow{k} x$. Toeplitz proved the following result which we leave as an exercise.

2.4. Exercise. (a) If the matrix (s_{kn}) satisfies the following conditions, called *Toeplitz conditions*:

(i) $s_{kn} \xrightarrow{k} 0, \ \forall n \in \mathbb{N}$,

(ii) $\sum\limits_{n=0}^{\infty} |s_{kn}| \le C < \infty, \ \forall k \in \mathbb{N}$,

(iii) $\sum\limits_{n=0}^{\infty} s_{kn} \xrightarrow{k} 1$,

then (s_{kn}) is regular.

(b) If (s_{kn}) satisfies only (i), (ii) and $x_n \to 0$, then $y_k \to 0$.

So it is enough to check that (i), (ii), (iii) hold for

$$s_{kn} = \left(\frac{\sin nh_k}{nh_k}\right)^2 - \left(\frac{\sin(n+1)h_k}{(n+1)h_k}\right)^2.$$

Clearly (i), (iii) hold. For (ii), let $u(x) = (\frac{\sin x}{x})^2$. Then we have

2.5. Exercise. $\int\limits_0^{\infty} |u'(x)| dx < \infty$.

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} |s_{kn}| &= \sum_{n=0}^{\infty} \left| \int_{nh_k}^{(n+1)h_k} u'(x) dx \right| \\ &\le \int_0^{\infty} |u'(x)| dx = C < \infty. \end{aligned}$$

$\dashv$

2.6. Riemann's Second Lemma. *Let $S \sim \sum c_n e^{inx}$ be a trigonometric series with $c_n \to 0$. Then*

$$\frac{\Delta^2 F_S(x,h)}{h} = \frac{F_S(x+h) + F_S(x-h) - 2F_S(x)}{h} \to 0,$$

as $h \to 0$, *uniformly* on x.

Proof. By direct calculation we have again

$$\frac{\Delta^2 F_S(x,2h)}{4h} = \sum \frac{\sin^2(nh)}{n^2 h} c_n e^{inx}$$

(where for $n=0$, $\frac{\sin^2(nh)}{n^2 h}$ is defined to be h). Let as before $0 < h_k \le 1$, $h_k \to 0$ and put

$$t_{kn} = \frac{\sin^2(nh_k)}{n^2 h_k}.$$

We have to show that $\sum(c_n e^{inx})t_{kn} \to 0$ as $k \to \infty$, uniformly on x. Since $c_n e^{inx} \to 0$, uniformly on x, it is enough to verify that (t_{kn}) satisfies the first two Toeplitz conditions (i), (ii), of 2.4.

Clearly (i) holds. To prove (ii) fix k and choose $N > 1$ with

$$N - 1 \le h_k^{-1} < N.$$

Then

$$\sum_{n=1}^{\infty} |t_{kn}| = \sum_{n=1}^{N-1} \frac{\sin^2(nh_k)}{n^2 h_k} + \sum_{n=N}^{\infty} \frac{\sin^2(nh_k)}{n^2 h_k}$$

$$\le (N-1)\cdot h_k + \frac{1}{h_k}\sum_{n=N}^{\infty}\frac{1}{n^2}$$

$$\le 1 + \frac{1}{h_k}\sum_{n=N}^{\infty}\frac{1}{n(n-1)}$$

$$= 1 + \frac{1}{h_k}\cdot\frac{1}{N-1} \le 3,$$

and we are done. ⊣

Note that this implies that the graph of F_S can have no corners, i.e., if the left- and right-derivatives of F_S exist at some point x, then they must be equal.

§3. The Cantor Uniqueness Theorem.

The following result was proved originally by Cantor with "set of positive measure" replaced by "interval".

3.1. The Cantor-Lebesgue Lemma. *If* $a_n \cos(nx) + b_n \sin(nx) \to 0$*, for* x *in a set of positive (Lebesgue) measure, then* $a_n, b_n \to 0$*. So if* $\sum c_n e^{inx} = 0$ *for* x *in a set of positive measure, then* $c_n \to 0$*.*

Proof. We can assume that $a_n, b_n \in \mathbb{R}$. Let $\rho_n = \sqrt{a_n^2 + b_n^2}$ and φ_n be such that

$$a_n \cos(nx) + b_n \sin(nx) = \rho_n \cos(nx + \varphi_n).$$

Thus $\rho_n \cos(nx + \varphi_n) \to 0$ on $E \subseteq [0, 2\pi)$, a set of positive measure. Assume $\rho_n \not\to 0$, toward a contradiction. So there is $\epsilon > 0$ and $n_0 < n_1 < n_2 < \cdots$ such that $\rho_{n_k} \geq \epsilon$. Then $\cos(n_k x + \varphi_{n_k}) \to 0$, so $2\cos^2(n_k x + \varphi_{n_k}) \to 0$, i.e., $1 + \cos 2(n_k x + \varphi_{n_k}) \to 0$ for $x \in E$. By Lebesgue Dominated Convergence $\int_E (1 + \cos 2(n_k x + \varphi_{n_k}))dx \to 0$, i.e., letting χ_E be the characteristic function of E in the interval $[0, 2\pi)$ extended with period 2π over all of $\mathbb{R}$, and $\mu(E)$ be the Lebesgue measure of E:

$$\mu(E) + \int_0^{2\pi} \chi_E(x) \cos 2(n_k x + \varphi_{n_k}) dx$$

$$= \mu(E) + 2\pi \left[\operatorname{Re} \hat{\chi}_E(-2n_k) \cdot \cos 2\varphi_{n_k} - \operatorname{Im} \hat{\chi}_E(-2n_k) \cdot \sin 2\varphi_{n_k} \right] \to 0.$$

We now have:

3.2. Exercise (Riemann-Lebesgue). If f is an integrable 2π-periodic function, then $\hat{f}(n) \to 0$ as $|n| \to \infty$. [*Remark.* This is really easy if f is the characteristic function of an interval.]

So it follows that $\mu(E) = 0$, a contradiction. ⊣

We are only one lemma away from the proof of Cantor's Theorem. This lemma was proved by Schwartz in response to a request by Cantor.

3.3 Lemma (Schwartz). *Let $F : (a, b) \to \mathbb{R}$ be continuous such that $D^2F(x) \geq 0$, $\forall x \in (a, b)$. Then F is convex on (a, b). In particular, if $F : (a, b) \to \mathbb{C}$ is continuous and $D^2F(x) = 0$, $\forall x \in (a, b)$, then F is linear on (a, b).*

Proof. By replacing F by $F + \epsilon x^2, \epsilon > 0$, and letting $\epsilon \to 0$, we can assume actually that $D^2F(x) > 0$ for all $x \in (a, b)$.

Assume F is not convex, toward a contradiction. Then there is a linear function $\mu x + \nu$ and $a < c < d < b$ such that if $G(x) = F(x) - (\mu x + \nu)$, then $G(c) = G(d) = 0$ and $G(x) > 0$ for some $x \in (c, d)$. Let x_0 be a point where G achieves its maximum in $[c, d]$. Then $c < x_0 < d$. Now for small enough h, $\Delta^2 F(x_0, h) \leq 0$, so $D^2F(x_0) \leq 0$, a contradiction. ⊣

We now have:

3.4. Theorem (Cantor, 1870). *If $\sum c_n e^{inx} = 0$ for all x, then $c_n = 0$, $\forall n \in \mathbb{Z}$.*

Proof. Let $S \sim \sum c_n e^{inx}$. By the Cantor-Lebesgue Lemma, $c_n \to 0$, as $|n| \to \infty$, so, in particular, c_n is bounded. By Riemann's First Lemma, $D^2F_S(x) = 0$, $\forall x \in \mathbb{R}$, and so by Schwartz's Lemma F_S is linear, i.e.,

$$c_0 \frac{x^2}{2} - \sum{}' \frac{1}{n^2} c_n e^{inx} = ax + b.$$

Put $x = \pi, x = -\pi$ and subtract to get $a = 0$. Put $x = 0, x = 2\pi$ and subtract to get $c_0 = 0$. Then $\sum' \frac{1}{n^2} c_n e^{inx} = b$. Since this series converges uniformly, if $m \neq 0$ we have

$$0 = \int_0^{2\pi} be^{-imx} dx$$

$$= \sum{}' \frac{1}{n^2} c_n \int_0^{2\pi} e^{i(n-m)x} dx$$

$$= \frac{c_m}{m^2},$$

so $c_m = 0$ and we are done. $\dashv$

Remark. Kronecker (when he was still on speaking terms with Cantor) pointed out that the use of the fact that $c_n \to 0$ was unnecessary, i.e., if one can prove that (1) $\sum c_n e^{inx} = 0,\ \forall x \in \mathbb{R}\ \&\ c_n \to 0 \Rightarrow c_n = 0, \forall n \in \mathbb{Z}$, then it follows that (2) $\sum c_n e^{inx} = 0, \forall x \in \mathbb{R} \Rightarrow c_n = 0, \forall n \in \mathbb{Z}$.

To see this assume (1) and take any series $\sum c_n e^{inx}$ such that $\sum c_n e^{inx} = 0, \forall x \in \mathbb{R}$ (without any other assumption on the c_n). Put for x, $x + u$ and $x - u$, add and divide by 2 to get

$$\sum c_n e^{inx} \cos nu = 0,\ \forall x, u \in \mathbb{R}.$$

or equivalently

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cos nu = 0, \forall x, u \in \mathbb{R}.$$

Considering x fixed now, we note that $a_n \cos nx + b_n \sin nx \to 0$ (since $c_0 + \sum (a_n \cos nx + b_n \sin nx) = 0$) so we can apply (1) to get that $a_n \cos nx + b_n \sin nx = 0$, for all $x, n \in \mathbb{N}$, from which easily $a_n = b_n = 0,\ \forall n \in \mathbb{N}$.

§4. Sets of uniqueness.

Cantor next extended his uniqueness theorem in 1871 by allowing a finite number of exceptional points.

4.1. Theorem (Cantor, 1871). *Assume that $\sum c_n e^{inx} = 0$ for all but finitely many $x \in \mathbb{R}$. Then $c_n = 0,\ \forall n \in \mathbb{Z}$.*

Proof. Let $S \sim \sum c_n e^{inx}$. Suppose $x_0 = 0 \leq x_1 < x_2 < \cdots < x_n < 2\pi = x_{n+1}$ are such that for $x \neq x_i$, $\sum c_n e^{inx} = 0$. Then, by Schwartz's Lemma, F_S is linear in each interval (x_i, x_{i+1}). Since by Riemann's Second Lemma the graph of F_S has no corners, it follows that F_S is linear in $[0, 2\pi)$. The same holds of course for any interval of length 2π, so F_S is linear and thus as in the proof of 3.4, $c_n = 0,\ \forall n \in \mathbb{Z}$. $\dashv$

In the sequel it will be convenient to identify the unit circle $\mathbb{T} = \{e^{ix} : 0 \leq x < 2\pi\}$ with $\mathbb{R}/2\pi\mathbb{Z}$ via the map $x \mapsto e^{ix}$. Often we can think of $\mathbb{T}$ as being $[0, 2\pi)$, or $[0, 2\pi]$ with $0, 2\pi$ identified. Functions on $\mathbb{T}$ can be also thought of as 2π-periodic functions on $\mathbb{R}$. We denote by λ the Lebesgue measure on $\mathbb{T}$ (i.e., the one induced by the above identification with $[0, 2\pi)$) normalized so that $\lambda(\mathbb{T}) = 1$.

We now introduce the following basic concept.

Definition. Let $E \subseteq \mathbb{T}$. We say that E is a *set of uniqueness* if every trigonometric series $\sum c_n e^{inx}$ which converges to 0 off E (i.e., $\sum c_n e^{inx} = 0$, for $e^{ix} \notin E$, which we will simply write "$x \notin E$") is identically 0. Otherwise it is called a set of *multiplicity.*

So 3.4 says that $\emptyset$ is a set of uniqueness and 4.1 says that every finite set is a set of uniqueness.

We denote by $\mathcal{U}$ the class of sets of uniqueness and by $\mathcal{M}$ the class of sets of multiplicity.

Our next goal is to prove the following extension of Cantor's Theorem.

4.2. Theorem (Cantor, Lebesgue – see Remarks in §6 below). *Every countable closed set is a set of uniqueness.*

We will give a proof using the method of transfinite induction.

§5. The Cantor-Bendixson Theorem.

Let $E \subseteq \mathbb{T}$ be a closed set. We define its *Cantor-Bendixson derivative* E' by

$$E' = \{x \in E : \ x \text{ is a limit point } of \ E\}.$$

Note that $E' \subseteq E$ and E' is closed as well.

Now define by transfinite induction for each ordinal α a closed set $E^{(\alpha)}$ as follows:

$$E^{(0)} = E,$$

$$E^{(\alpha+1)} = (E^{(\alpha)})'$$

$$E^{(\lambda)} = \bigcap_{\alpha<\lambda} E^{(\alpha)}, \ \lambda \text{ a limit ordinal.}$$

The $E^{(\alpha)}$ form a decreasing sequence of closed sets contained in E:

$$E^{(0)} \supseteq E^{(1)} \supseteq E^{(2)} \supseteq \cdots \supseteq E^{(\alpha)} \supseteq \cdots \supseteq E^{(\beta)} \supseteq \cdots, \alpha \leq \beta.$$

5.1. Lemma. *If F_α, α an ordinal, is a decreasing sequence of closed sets, then for some countable ordinal α_0 we have that*

$$F_{\alpha_0} = F_{\alpha_0+1}.$$

Proof. Fix a basis $\{U_n\}$ for the topology of $\mathbb{T}$ and let

$$A_\alpha = \{n : U_n \cap F_\alpha = \emptyset\}.$$

Then $\alpha \leq \beta \Rightarrow A_\alpha \subseteq A_\beta$. If $A_\alpha \subsetneqq A_{\alpha+1}$ for all countable ordinals α, let $f(\alpha)$ be the least n with $n \in A_{\alpha+1} \setminus A_\alpha$. Then, if ω_1 is the first uncountable ordinal, which by standard practice we also identify with the set $\{\alpha : \alpha < \omega_1\}$ of all countable ordinals, we have that $f : \omega_1 \to \mathbb{N}$ is injective, which is a contradiction. So for some $\alpha_0 < \omega_1$, $A_{\alpha_0} = A_{\alpha_0+1}$, so $F_{\alpha_0} = F_{\alpha_0+1}$. ⊣

Thus for each closed set $E \subseteq \mathbb{T}$, there is a least countable α with $E^{(\alpha)} = E^{(\alpha+1)}$ and thus $E^{(\alpha)} = E^{(\beta)}$ for all $\alpha \leq \beta$. We denote this ordinal by $rk_{\mathrm{CB}}(E)$ and call it the *Cantor-Bendixson rank* of E. We also put

$$E^{(\infty)} = E^{(rk_{\mathrm{CB}}(E))}.$$

Notice that $(E^{(\infty)})' = E^{(\infty)}$, so $E^{(\infty)}$ is perfect, i.e., every point of it is a limit point (it could be $\emptyset$ though). We call it the *perfect kernel* of E.

5.2. Exercise. Show that $E^{(\infty)}$ is the largest perfect set contained in E.

5.3. Theorem (Cantor-Bendixson). *Let E be closed. Then the set $E \setminus E^{(\infty)}$ is countable. In particular,*

$$E \textit{ is countable} \Leftrightarrow E^{(\infty)} = \emptyset.$$

Proof. Let $x \in E \setminus E^{(\infty)}$, so that for some (unique) $\alpha < rk_{\mathrm{CB}}(E), x \in E^{(\alpha)} \setminus E^{(\alpha+1)}$. Since there are only countably many such α, it is enough to prove the following:

5.4. Lemma. *For any closed set $F, F \setminus F'$ is countable.*

Proof. Fix a countable basis $\{U_n\}_{n\in\mathbb{N}}$. If $x \in F \setminus F'$, then there is some n with $F \cap U_n = \{x\}$. So $F \setminus F' = \bigcup\{F \cap U_n : F \cap U_n$ is a singleton$\}$, which is clearly countable. ⊣

5.5. Exercise. Show that for each closed set E there is a unique decomposition $E = P \cup C$, $P \cap C = \emptyset$, P perfect, C countable.

5.6. Exercise. For each countable successor ordinal α find a countable closed set E with $rk_{\mathrm{CB}}(E) = \alpha$.

§6. Sets of uniqueness (cont'd).

We are now ready to give the

Proof of Theorem 4.2.

Let E be a countable closed set. Let $S \sim \sum c_n e^{inx}$ be such that $\sum c_n e^{inx} = 0$ off E. Since it is clear that any translate (in $\mathbb{T}$) of a set of uniqueness is also a set of uniqueness, we can assume that $0 \notin E$. So we can view E as being a closed set contained in $(0, 2\pi)$. The complement in $(0, 2\pi)$ of any closed subset F of $(0, 2\pi)$ is a disjoint union of open intervals with endpoints in $F \cup \{0, 2\pi\}$, called its *contiguous intervals.* We will prove by transfinite induction on α, that F_S is linear on each contiguous interval of $E^{(\alpha)}$. Since $E^{(\alpha_0)} = \emptyset$ for some α_0, it follows that F_S is linear on $(0, 2\pi)$, so, as before, $c_n = 0, \forall n$.

$\alpha = 0$: This is clear since $\sum c_n e^{inx} = 0$ on each contiguous interval of $E^{(0)} = E$.

$\alpha \Rightarrow \alpha+1$: Assume F_S is linear in each contiguous interval of $E^{(\alpha)}$. Let then (a, b) be a contiguous interval of $E^{(\alpha+1)}$. Then in each closed subinterval $[c, d] \subseteq (a, b)$ there are only finitely many points $c \leq x_0 < x_1 < \cdots < x_n \leq d$ of $E^{(\alpha)}$. Then $(c, x_0), (x_0, x_1), \cdots, (x_n, d)$ are contained in contiguous intervals of $E^{(\alpha)}$, so, by induction hypothesis, F_S is linear in each one of them, so by the Riemann Second Lemma again, F_S is linear on $[c, d]$ and thus on (a, b).

$\alpha < \lambda \Rightarrow \lambda$ (λ a limit ordinal): We use a compactness argument. Fix a contiguous interval (a, b) of $E^{(\lambda)}$ and a closed subinterval $[c, d] \subseteq (a, b)$. Then

$$
\begin{aligned}
[c, d] &\subseteq (0, 2\pi) \setminus E^{(\lambda)} \\
&= (0, 2\pi) \setminus \bigcap_{\alpha<\lambda} E^{(\alpha)} \\
&= \bigcup_{\alpha<\lambda} [(0, 2\pi) \setminus E^{(\alpha)}].
\end{aligned}
$$

Since $(0, 2\pi) \setminus E^{(\alpha)}$ is open and $[c, d]$ is compact, there are finitely many $\alpha_1, \cdots, \alpha_n < \lambda$ with

$$[c, d] \subseteq \bigcup_{\alpha \in \{\alpha_1, \cdots, \alpha_n\}} [(0, 2\pi) \setminus E^{(\alpha)}] \subseteq (0, 2\pi) \setminus E^{(\beta)}$$

for any $\alpha_1, \cdots, \alpha_n \leq \beta < \lambda$. Then $[c, d]$ is contained in a contiguous interval of $E^{(\beta)}$, so, by induction hypothesis, F_S is linear on $[c, d]$ and thus on (a, b).

This completes the proof. $\dashv$

Remark. Cantor published in 1872 the proof of Theorem 4.2 only for the case $rk_{\mathrm{CB}}(E) < \omega$, i.e., when the Cantor-Bendixson process terminates in finitely many steps. Apparently at this stage he had, at least at some intuitive level, the idea of extending this process into the transfinite at levels $\omega, \omega + 1, \cdots$. However this involved conceptual difficulties which led him to re-examine the foundations of the real number system and eventually to create set theory, including, several years later, the rigorous development of the theory of ordinal numbers and transfinite induction. However, after 1872 Cantor never returned to the problem of uniqueness and never published a complete proof of 3.5. This was done, much later, by Lebesgue in 1903.

Theorem 4.1 was further extended by Bernstein (1908) and W. H. Young (1909) to show that an *arbitrary* countable set is a set of uniqueness. Finally in 1923 Bari showed that the union of countably many *closed* sets of uniqueness is a set of uniqueness.

6.1. Exercise (Bernstein, Young). Assume these results and the fact that any uncountable Borel set contains a non-$\emptyset$ perfect subset. Show that every set which contains no perfect non-$\emptyset$ set is a set of uniqueness.

PART II. BAIRE CATEGORY METHODS.

§7. Sets of uniqueness and Lebesgue measure.

To get some idea about the size of sets of uniqueness, we will first prove the following easy fact.

7.1. Proposition. *Let $A \subseteq \mathbb{T}$ be a (Lebesgue) measurable set of uniqueness. Then A is null, i.e., $\lambda(A) = 0$.*

Proof. Assume $\lambda(A) > 0$, towards a contradiction. Then, by regularity, $\lambda(F) > 0$ for some *closed* subset $F \subseteq A$. Consider the characteristic function χ_F of F and its Fourier series $S(\chi_F)$. We need the following standard fact.

7.2. Lemma (Localization principle for Fourier series). *Let f be an integrable function on $\mathbb{T}$. Then the Fourier series of f converges to 0 in any open interval in which f vanishes.*

Proof. We have

$$
\begin{aligned}
S_N(f, x) &= \sum_{-N}^{N} \hat{f}(n) e^{inx} \\
&= \sum_{-N}^{N} \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt\right) e^{inx} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(t) \left(\sum_{-N}^{N} e^{in(x-t)} \right) dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(t) D_N(x - t) dt \ (= f * D_n(x)),
\end{aligned}
$$

where the *Dirichlet kernel* D_n is defined by

$$D_n(u) = \sum_{-N}^{N} e^{inu} = \frac{\sin(n + \frac{1}{2})u}{\sin \frac{u}{2}} = \cos nu + \cot(\frac{u}{2}) \sin nu.$$

So (changing $\int_0^{2\pi}$ to $\int_{-\pi}^{\pi}$)

$$S_N(f, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cot(\frac{t}{2}) \sin nt dt.$$

If now f vanishes in an interval around 0, then clearly $f(t) \cot(\frac{t}{2})$ is integrable, so these two integrals converge to 0 by the Riemann-Lebesgue Lemma (3.2). So $S_N(f, 0) \to 0$.

Thus we have shown that if f vanishes in an interval around 0, then its Fourier series converges to 0 at 0. By translation this is true for any other point. $\dashv$

So $S(\chi_F)$ converges to 0 in any interval disjoint from F, i.e., $S(\chi_F)$ converges to 0 off F. So, since F is a set of uniqueness, $\hat{\chi}_F(n) = 0$ for all n. But $\hat{\chi}_F(0) = \lambda(F) > 0$, a contradiction. $\dashv$

7.3. Exercise (Bernstein). Show that there is $A \subseteq \mathbb{T}$ such that neither A nor $\mathbb{T} \setminus A$ contain a non-$\emptyset$ perfect set (such a set is called a *Bernstein* set). Conclude that A is not measurable and therefore that A is a set of uniqueness which is not null.

So we have

$$\text{countable } \subseteq \mathcal{U} \cap \text{ measurable } \subseteq \text{ null}.$$

In the beginning of the century it was widely believed that $\mathcal{U} \cap$ measurable = null, i.e., if a trigonometric series $\sum c_n e^{inx}$ converges to 0 almost everywhere (a.e.), then it is identically 0. Recall, for example, here the following standard fact (which we will not use later on).

7.4. Theorem. (Fejér-Lebesgue). *For every integrable function f on $\mathbb{T}$, let*

$$\sigma_N(f, x) = \frac{S_0(f, x) + S_1(f, x) + \cdots + S_N(f, x)}{N + 1}$$

be the average of the partial sums $S_N(f, x)$ of the Fourier series of f. Then $\sigma_N(f, x) \to f(x)$ a.e. In particular, if a Fourier series converges a.e. to 0, it is identically 0.

For example, in Luzin's dissertation *Integration and trigonometric series* (1915), this problem is discussed and is considered improbable that there is a non-zero trigonometric series that converges to 0 a.e. It thus came as a big surprise when in 1916 Menshov proved that there are indeed trigonometric series which converge to 0 a.e. but are not identically 0. It follows, for example, that any function which admits a trigonometric expansion admits actually more than one, if only convergence a.e. to the function is required.

Remark. Menshov also showed that *every* measurable function admits an a.e. trigonometric expansion, i.e., for any 2π-periodic measurable f there is a trigonometric series $\sum c_n e^{inx}$ such that $f(x) = \sum c_n e^{inx}$ a.e. By the above this series is *not* unique.

Menshov actually constructed an example of a *closed* set of multiplicity of measure 0, by an appropriate modification of the standard construction of the Cantor set. The Cantor set in the interval $[0, 2\pi]$ is constructed by removing the middle 1/3 open interval, then in each of the remaining two closed intervals remaining the middle 1/3 open interval, etc. Now suppose we modify the construction by removing in the first stage the middle 1/2 interval, at the second stage, the middle 1/3 interval, at the third stage the middle 1/4 interval, etc. Denote the resulting perfect set by E_M.

7.5. Exercise. Show that $\lambda(E_M) = 0$.

Now Menshov showed that E_M is a set of multiplicity as follows: This set is in a canonical 1-1 correspondence with the set of infinite binary sequences $2^{\mathbb{N}}$. Take the usual coin-tossing measure on $2^{\mathbb{N}}$ and transfer it by this correspondence to E_M. Call this measure μ_M. (Thus μ_M gives equal measure $1/2^n$ to each of the 2^n closed intervals at the nth stage of this construction.) It is a probability Borel measure on $\mathbb{T}$, so we can define its *Fourier(-Stieltjes) coefficients* $\hat{\mu}_M(n)$, as usual, by

$$\hat{\mu}_M(n) = \int e^{-int} d\mu_M(t).$$

Then it turns out that $\sum \hat{\mu}_M(n)e^{inx} = 0$ for $x \notin E$, so E is a set of multiplicity (as $\hat{\mu}_M(0) = \mu(E) > 0$). This might seem not too surprising, since after all the measure μ "lives" on E_M. However by the same token one should also expect that if we consider the usual Cantor set E_C and denote the corresponding measure by μ_C we should have $\sum \hat{\mu}_C(n)e^{inx} = 0$ off the Cantor set as well, i.e., the Cantor set should also be a set of multiplicity, which is false! This is only one of the many phenomena in this subject which challenge your intuition.

What is the difference between μ_M, μ_C that accounts for this phenomenon? It turns out that it is the following: $\hat{\mu}_M(n) \to 0$ but $\hat{\mu}_C(n) \not\to 0$ as $|n| \to \infty$. In fact, we have the following result.

7.6. Theorem. *Let $E \subseteq \mathbb{T}$, $E \neq \mathbb{T}$ be a closed set and let μ be a probability Borel measure on $\mathbb{T}$ with $\mu(E) = 1$. Then the following are equivalent:*

(i) $\hat{\mu}(n) \to 0$;

(ii) $\sum \hat{\mu}(n)e^{inx} = 0$, $\forall x \notin E$.

Clearly (ii) $\Rightarrow$ (i) by the Cantor-Lebesgue Lemma. The proof of (i) $\Rightarrow$ (ii) requires some further background in the theory of trigonometric series and we will postpone it for a while (see §§12,13).

Thus Menshov's proof is based on the fact that $\hat{\mu}_M(n) \to 0$, which is proved by a delicate calculation.

We will develop in the sequel a totally different approach to Menshov's Theorem on the existence of null sets of multiplicity, an approach based on the Baire category method.

§8. Baire category.

A set in a topological space is *nowhere dense* if its closure has empty interior. A set is first *category* or *meager* if it is contained in a countable union of nowhere dense sets. Otherwise it is of the *second category* or *non-meager.* It is clear that meager sets form a σ-ideal, i.e., are closed under subsets and countable unions. So this concept determines a notion of "*topological smallness*" analogous to that of null sets in measure theory. Of course, for this to be of any interest it better be that it doesn't trivialize, i.e., that the whole space is not meager. This is the case in well-behaved spaces, like complete metric spaces. In fact we have an even stronger statement known as the Baire Category Theorem (for such spaces). We call a set *comeager* if its complement is meager. Notice that a set is comeager iff it contains a countable intersection of dense open sets.

8.1 Theorem (The Baire Category Theorem). *Let X be a complete metric space. Every comeager set is dense (in particular non-$\emptyset$).*

This is the basis of a classical method of existence proof in mathematics: Suppose we want to show the existence of a mathematical object x satisfying some property P. The *category method* consists of finding an appropriate complete metric space X (or other "nice" topological space satisfying the Baire Category Theorem) and showing that $\{x \in X : P(x)\}$ is comeager in X. This not only shows that $\exists x P(x)$, but in fact that in the space X "most" elements of X have property P or as it is often expressed the "*generic*" element of X satisfies property P. (Similarly, if a property holds a.e. we say that the "*random*" element satisfies it.)

A standard application of the Baire Category Theorem is Banach's proof of the existence of continuous nowhere differentiable functions (originally due to Bolzano and Weierstrass in the 19th century). The argument goes as follows:

Let $C(\mathbb{T})$ be the space of real valued continuous 2π-periodic functions with the uniform (or sup) metric

$$d(f, g) = \sup\{\, |f(x) - g(x)| : \ x \in \mathbb{T}\}.$$

It is well-known that this is a complete metric space. We want to show that

$$\{f \in C(\mathbb{T}) : \ \forall x (f'(x) \text{ does not exist})\}$$

is comeager in $C(\mathbb{T})$, so non-$\emptyset$. Consider, for each n, the set

$$U_n = \{f \in C(\mathbb{T}) : \ \forall x \exists h > 0 \left| \frac{f(x+h) - f(x)}{h} \right| > n\}.$$

8.2. Exercise. Show that U_n is open in $C(\mathbb{T})$.

It is not hard to show now that U_n is also dense in $U(\mathbb{T})$. Simply approximate any $f \in C(\mathbb{T})$ by a piecewise linear function g and then approximate g by a piecewise linear function with big slopes, so that it belongs to U_n.

Thus $\bigcap_n U_n$ is a countable intersection of dense open sets, so it is comeager. But clearly if $f \in \bigcap_n U_n$, f is nowhere differentiable. (So the "generic" continuous function is nowhere differentiable.)

Before we proceed, let us recall that a set A, in a topological space X, is said to have the *Baire property* (BP) if there is an open set U such that $A \Delta U$ is meager. The class of sets with the BP is a σ-algebra (i.e., is closed under countable unions and complements), in fact it is the smallest σ-algebra containing the open sets and the meager sets. The sets with the BP are analogs of the measurable sets.

Although there is some analogy between category and measure it should be emphasized that the concepts are "orthogonal". This can be expressed by the following important fact:

8.3. Proposition. *There is a dense G_δ (so comeager) set $G \subseteq \mathbb{T}$ such that $\lambda(G) = 0$ (i.e., G is null).*

Proof. Let $\{d_n\}$ be a countable dense set in $\mathbb{T}$. For each m let $I_{n,m}$ be an open interval around d_n with $\lambda(I_{n,m}) \leq \frac{1}{m} \cdot 2^{-n}$. Let $U_m = \bigcup_n I_{n,m}$, so that U_m is dense open with $\lambda(U_m) \leq \sum_n \lambda(I_{n,m}) \leq 2/m$. Let $G = \bigcap_m U_m$. ⊣

§9. Sets of uniqueness and category.

We have seen in §7 that a (measurable) set of uniqueness is measure theoretically negligible, i.e., null. The question was raised, already in the 1920's (see, e.g., the memoir of N. Bari in Fundamenta Mathematica, Bari [1927]) whether they are also topologically negligible, i.e., meager (assuming they have the BP). So we have

9.1. The Category Problem. Is every set of uniqueness with the BP of the first category?

This problem was solved affirmatively by Debs and Saint Raymond in 1986. Their original proof used the descriptive set theoretic methods that we will discuss in Part III and was quite sophisticated, making use of machinery established in earlier work of Solovay, Kaufman, Kechris-Louveau-Woodin and Kechris-Louveau. In fact Debs and Saint Raymond established the following stronger result.

9.2. Theorem (Debs-Saint Raymond). *Let $A \subseteq \mathbb{T}$ be a non-meager set with the BP. Then there is a Borel probability measure μ on $\mathbb{T}$ with $\mu(A) = 1$ and $\hat{\mu}(n) \to 0$, as $|n| \to \infty$.*

To see that this implies 9.1 we argue as follows:

Let A be a set of uniqueness with the BP. If A is not meager, there is μ, a Borel probability measure, with $\mu(A) = 1$, and $\hat{\mu}(n) \to 0$. Since every Borel probability measure is regular, there is closed $F \subseteq A$ with $\mu(F) > 0$. Let $\nu = \mu|F$, i.e., $\nu(X) = \mu(X \cap F)$.

9.3. Proposition. *$\hat{\nu}(n) \to 0$, as $|n| \to \infty$.*

Proof. We have for any continuous function on $\mathbb{T}$,

$$\int f d\nu = \int f \chi_F d\mu,$$

so

$$\hat{\nu}(n) = \int \chi_F(t) e^{-int} d\mu(t).$$

For each $\epsilon > 0$, there is a trigonometric polynomial $P(x) = \sum_{-N}^{N} c_k e^{ikx}$ such that $\int |\chi_F - P| d\mu < \epsilon$. Now if

$$d_n = \int P(t) e^{-int} d\mu(t),$$

we have

$$d_n = \int \Big(\sum_{-N}^{N} c_k e^{ikx} \Big) e^{-int} d\mu(t)$$

$$= \sum_{k=-N}^{N} c_k \hat{\mu}(n-k) \to 0$$

as $|n| \to \infty$. Moreover

$$|\hat{\nu}(n) - d_n| = \left| \int (\chi_F(t) - P(t)) e^{-int} d\mu \right|$$

$$\leq \int |\chi_F - P| d\mu < \epsilon,$$

so $\overline{\lim}_n |\hat{\nu}(n)| \leq \epsilon$, and thus $\hat{\nu}(n) \to 0$. $\dashv$

Then we can apply 7.6 to conclude that $\sum \hat{\nu}(n)e^{inx} = 0$ off E and thus off A. As $\hat{\nu}(0) \neq 0$, this shows that A is a set of multiplicity, a contradiction.

Starting with the next section we will give a very different than the original and much simpler proof of 9.2, due to Kechris-Louveau, which is based on the category method and employs only elementary functional analysis. Before we do that though we want to show how 9.2 gives also a much different proof of Menshov's Theorem, which "explains" this result as an instance of the "orthogonality" of the concepts of null and meager sets.

Indeed, by 8.3 fix a dense G_δ set $G \subseteq \mathbb{T}$ with $\lambda(G) = 0$. Then G is comeager, so by 9.2, there is a Borel probability measure μ with $\mu(G) = 1$ and $\hat{\mu}(n) \to 0$. As before, find $F \subseteq G$ closed with $\mu(F) > 0$. Then $\sum \hat{\mu}(n)e^{inx} = 0$ off F and thus off G and this shows that $\sum \hat{\mu}(n)e^{inx}$ converges to 0 a.e. without being identically 0 (it also shows that F is a null closed set of multiplicity, as Menshov also showed).

But this method has also many other applications. For example, in the 1960's Kahane and Salem raised the following question: Recall that a number $x \in [0, 2\pi]$ is called *normal* in base 2 (say) if for $\frac{x}{2\pi} = 0 \cdot x_1 x_2 \cdots$, $x_i \in \{0, 1\}$, and all $(a_0, \cdots, a_{k-1})$, $a_i \in \{0, 1\}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \text{card}\,\{1 \leq m \leq n : x_{m+i} = a_{m+i}, 0 \leq i \leq k-1\} = \frac{1}{2^k}.$$

Denote by N the set of normal numbers and by N' its complement. A famous theorem of Borel asserts that almost every number is normal, i.e., $\lambda(N) = 1$, so $\lambda(N') = 0$. Kahane and Salem asked whether N' supports any probability Borel measure μ (i.e., $\mu(N') = 1$) with $\hat{\mu}(n) \to 0$. (Measures with $\hat{\mu}(n) \to 0$ are somehow considered "thick" - recall that $\hat{\lambda}(n) \to 0$, since in fact $\hat{\lambda}(n) = 0$ if $n \neq 0$. It can be also shown that they are continuous, i.e., give every singleton measure 0. In fact a theorem of Wiener asserts that $\sum\limits_{x\in\mathbb{T}} \mu(\{x\})^2 = \lim\limits_{n\to\infty} \frac{1}{2N+1} \sum\limits_{-N}^{N} |\hat{\mu}(n)|^2$.) Lyons, in 1983, answered this affirmatively, by using delicate analytical tools. However, a totally different proof can be based on 9.2 by noticing that

9.4. Proposition. *N' is comeager.*

Proof. We show that N is meager. Note that

$$N \subseteq \bigcup_{n\geq 1} F_n,$$

where

$$F_n = \bigcap_{k \geq n} F'_k,$$

with

$$F'_k = \left\{ 2\pi \sum_{l=1}^{\infty} x_l 2^{-l} \;:\; x_l = 0, 1 \text{ and } \left| \frac{x_1 + \cdots x_k}{k} - \frac{1}{2} \right| \leq \frac{1}{4} \right\}.$$

Now F'_k is closed being the continuous image of the closed (thus compact) subset of $2^{\mathbb{N}}$

$$\left\{ (x_1, x_2, \cdots) \in 2^{\mathbb{N}} : \left| \frac{x_1 + \cdots + x_k}{k} - \frac{1}{2} \right| \leq \frac{1}{4} \right\},$$

by the continuous map

$$(x_1, x_2, \cdots) \mapsto 2\pi \sum_{l=1}^{\infty} x_l 2^{-l}.$$

So F_n is closed.

9.5. Exercise. F_n contains no open interval.

So each F_n is nowhere dense and N is meager. $\dashv$

§10. Review of duality in Banach spaces.

Good references for the basic results of functional analysis and measure theory that we will use in the sequel are Rudin [1973], [1987].

Let X be a Banach space, i.e., a complete normed linear space, over the complex numbers. Denote by X^* the dual space of X, i.e., the Banach space of all (bounded or, equivalently, continuous) linear functionals $x^* : X \to \mathbb{C}$ with the norm

$$||x^*|| = \sup \left\{ \frac{||x^*(x)||}{||x||} : x \in X, x \neq 0 \right\}$$

$$= \sup \left\{ ||x^*(x)|| : x \in X, ||x|| \leq 1 \right\}.$$

It is often convenient to write

$$x^*(x) = \langle x, x^* \rangle = \langle x^*, x \rangle.$$

A set $A \subseteq X$ is called *convex* if for every $x, y \in A$ and $t \in [0,1], tx + (1-t)y \in A$.

There is a fundamental collection of results, collectively known as the *Hahn-Banach Theorems,* which assert the existence of appropriate linear functionals. We will need here the following separation form of Hahn-Banach.

10.1. Theorem (Hahn-Banach). *Let X be a Banach space and let $A, B \subseteq X$ be convex, nonempty, $A \cap B = \emptyset$, with A compact and B closed. Then there is $x^* \in X^*$, and $\alpha, \beta \in \mathbb{R}$ with*

$$\text{Re } \langle x^*, x\rangle \ < \alpha < \beta < \ \text{Re } \langle x^*, y\rangle,$$

for $x \in A, y \in B$.

We now define the weak-topology of X as follows: It is the smallest topology on X for which the maps

$$x \mapsto \langle x^*, x\rangle, \text{ for } x^* \in X^*,$$

are continuous. It is thus contained in the usual or *norm-topology* of X, i.e., the one induced by its norm, and unless X is finite-dimensional, it is properly contained. By definition, a subbasis of this topology consists of all sets of the form

$$\{x : \langle x^*, x\rangle \in U\}$$

for $x^* \in X,\ U \subseteq \mathbb{C}$ open.

10.2. Exercise. Show that X with the weak-topology is a topological vector space (i.e., scalar multiplication and vector addition are continuous) and that a local basis at 0 is given by the sets $U_{x_1^*,\cdots,x_n^*,\epsilon} = \{x \in X : |\langle x_1^*, x\rangle|, \cdots, |\langle x_n^*, x\rangle| < \epsilon\}$, for $x_1^*, \cdots, x_n^* \in X^*$.

We will denote by $\overline{A}^w$ the closure of $A \subseteq X$ in the weak-topology. Clearly $\overline{A}$ (= the closure of A in the norm-topology) $\subseteq \overline{A}^w$, since there are more norm closed sets than weak closed sets. However, for convex sets these closures coincide.

10.3. Theorem (Mazur). *Let X be a Banach space. For every convex set $A \subseteq X$, $\overline{A} = \overline{A}^w$.*

Proof. It is enough to show that $\overline{A}^w \subseteq \overline{A}$. Let $x_0 \notin \overline{A}$ in order to show that $x_0 \notin \overline{A}^w$. By Hahn-Banach applied to $\{x_0\}, \overline{A}$ (which is easily convex) there is $x^* \in X^*$ and $\alpha \in \mathbb{R}$ with

$$\text{Re } \langle x^*, x_0\rangle < \alpha < \ \text{Re } \langle x^*, y\rangle$$

for $y \in \overline{A}$. Then $\{x : \text{Re}\langle x^*, x\rangle < \alpha\}$ is a weak-nbhd of x_0 which is disjoint from A, so $x_0 \notin \overline{A}^w$. $\dashv$

On the dual Banach space X^* we of course have its weak-topology but we can also consider an even weaker (fewer open sets) topology called the *weak*-topology* or *w^*-topology.* This is the smallest topology for which the functions

$$x^* \mapsto \langle x, x^*\rangle$$

for $x \in X$ are continuous. Since every $x \in X$ gives rise to a linear functional $x^{**} \in X^{**}$, defined by

$$\langle x^{**}, x^*\rangle = \langle x, x^*\rangle \text{ for } x^* \in X^*,$$

this shows that the weak*-topology of X^* is contained in its weak-topology (in general properly).

10.4. Exercise. Verify that X^* with the weak*-topology is a topological vector space and that a local basis at 0 is given by the sets $V_{x_1,\cdots,x_n,\epsilon} = \{x^* \in X : \ |\langle x_1, x^*\rangle|, \cdots, |\langle x_n, x^*\rangle| < \epsilon\}$ for $x_1, \cdots, x_n \in X$.

The crucial property of the weak*-topology is given by the following.

10.5. Theorem (Banach-Alaoglu). *Let X be a Banach space and consider a closed ball*

$$B_r(X^*) = \{x^* \in X^* : \ ||x^*|| \leq r\}$$

of X^*. Then $B_r(X^*)$ is weak*-compact, i.e., compact in the weak*-topology.

Proof. It is enough to consider $r = 1$. Consider the product space $\prod_{x \in X} \Delta_{||x||}$, where $\Delta_r = \{z \in \mathbb{C} : \ |z| \leq r\}$. This is compact, by Tychonoff's Theorem. Note that $B_1(X^*) \subseteq \prod_{x \in X} \Delta_{||x||}$, since $|x^*(x)| \leq ||x^*|| \cdot ||x|| \leq ||x||$. Moreover, the relative topology that $B_1(X^*)$ inherits from $\prod_{x \in X} \ \Delta_{||x||}$ is exactly the weak*-topology. So it is enough to show that $B_1(X^*)$ is closed in $\prod_{x \in X} \ \Delta_{||x||}$. Clearly $B_1(X^*) = \{f \in \prod_{x \in X} \Delta_{||x||} : \ f$ is linear$\} = \bigcap_{\alpha,\beta \in \mathbb{C}} \bigcap_{x,y \in X} \{f : \ f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)\}$. Since the map $f \mapsto f(x)$ from $\prod_{x \in X} \ \Delta_{||x||}$ into $\Delta_{||x||}$ is continuous, $B_1(X^*)$ is the intersection of closed sets, thus closed. ⊣

10.6. Exercise. Show that when X is separable (i.e., has a countable dense set), then $B_r(X^*)$ with the weak*-topology is also metrizable with compatible metric

$$d(x^*, y^*) = \sum 2^{-n} |\langle x_n, x^*\rangle - \langle y_n, y^*\rangle|,$$

where $\{x_n\}$ is dense in the unit ball $B_1(X)$ of X.

Remark. Every element x of X can be identified with the element x^{**} of $X^{**} = (X^*)^*$ given by

$$\langle x^{**}, x^*\rangle = \langle x, x^*\rangle.$$

So we can view X as a subset of X^{**} (it is in fact a closed subspace of X^*). It is obvious from the definition that the weak-topology on X is exactly the same as the weak*-topology of X, when it is viewed as a subset of $X^{**}(= (X^*)^*)$.

We will now discuss some important, and crucial for our purposes, examples.

A) First, we denote by $c_0 = c_0(\mathbb{Z})$ the Banach space of all sequences $(x_n)_{n \in \mathbb{Z}}, x_n \in \mathbb{C}$ such that $x_n \to 0$ as $|n| \to \infty$, equipped with the sup norm

$$||(x_n)||_\infty = \sup_{n \in \mathbb{Z}} |x_n|.$$

We denote by $\ell^1 = \ell^1(\mathbb{Z})$ the Banach space of all sequences $(x_n)_{n \in \mathbb{Z}}$ such that $\sum |x_n| < \infty$ with the norm

$$||(x_n)||_1 = \sum_{n \in \mathbb{Z}} |x_n|.$$

Finally, we denote by $\ell^\infty = \ell^\infty(\mathbb{Z})$ the Banach space of all bounded sequences $(x_n)_{n\in\mathbb{Z}}$ (i.e., $\sup_{n\in\mathbb{Z}}|x_n| < \infty$) with the sup norm

$$||(x_n)||_\infty = \sup_{n\in\mathbb{Z}} |x_n|$$

10.7. Exercise. Show that c_0 is a closed (linear) subspace of ℓ^∞. Show that c_0, ℓ^1 are separable, but ℓ^∞ is not.

We will now identify $c_0^*, (\ell^1)^*$. Let $\Lambda \in c_0^*$ and put $\lambda_n = \Lambda(e_{-n})$, where

$$e_n(i) = \begin{cases} 0, & \text{if } i \neq n \\ \\ 1, & \text{if } i = n \end{cases}, \text{ for } n, i \in \mathbb{Z}.$$

(The use of $\Lambda(e_{-n})$ instead of $\Lambda(e_n)$ is for technical convenience and assures consistency with the definition of Fourier coefficients $\hat{\mu}(n) = \int e^{-int} d\mu(t)$ later on.)

10.8. Exercise. For any $(x_n) \in c_0$,

$$\Lambda((x_n)) = \sum \lambda_n x_{-n},$$

$\sum |\lambda_n| < \infty$ and $||\Lambda|| = ||(\lambda_n)||_1$.

From this it immediately follows that the bijection

$$\Lambda \leftrightarrow (\lambda_n)$$

is a Banach space isomorphism between ℓ^1 and c_0^*, so we simply identify c_0^* with ℓ^1,

$$c_0^* = \ell^1,$$

and view every element of $\ell^1, (\lambda_n)$, as operating on an element (x_n) of c_0, by

$$\langle (\lambda_n), (x_n) \rangle = \langle (x_n), (\lambda_n) \rangle = \sum \lambda_n x_{-n}.$$

Now consider $(\ell^1)^*$ and put, as before, $\lambda_n = \Lambda(e_{-n})$.

10.9. Exercise. For any $(x_n) \in \ell^1$,

$$\Lambda((x_n)) = \sum \lambda_n x_{-n},$$

$\sup |\lambda_n| < \infty$ and $||\Lambda|| = \sup |\lambda_n|$.

So, as before, we can identify $(\ell^1)^*$ with ℓ^∞,

$$(\ell^1)^* = \ell^\infty$$

and view any element $(\lambda_n) \in \ell^\infty$ as operating on $(x_n) \in \ell^1$ by

$$\langle (\lambda_n), (x_n) \rangle = \langle (x_n), (\lambda_n) \rangle = \sum \lambda_n x_{-n}.$$

Note that as $c_0 \subseteq \ell^\infty$ there are two meanings of $\langle (\lambda_n), (x_n) \rangle$ for $(\lambda_n) \in c_0$, $(x_n) \in \ell^1$, but both give, of course, the same value.

10.10. Exercise. Let (x_n) be a bounded sequence of elements of c_0, i.e., $\sup ||x_n||_\infty < \infty$. Let $x \in c_0$. Show that $x_n \to x$ in the weak-topology iff $x_n(i) \to x(i)$, $\forall i \in \mathbb{Z}$. Show that if (x_n) is a bounded sequence in ℓ^1, and $x \in \ell^1$, then $x_n \to x$ in the weak*-topology iff $x_n(i) \to x(i)$, $\forall i \in \mathbb{Z}$. Finally, if (x_n) is a bounded sequence in ℓ^∞ and $x \in \ell^\infty$, then $x_n \to x$ in the weak*-topology (of $\ell^\infty = (\ell^1)^*$) iff $x_n(i) \to x(i)$, $\forall i \in \mathbb{Z}$.

B) Now consider the Banach space $C(\mathbb{T})$ of all continuous (complex) functions on $\mathbb{T}$ with the sup norm

$$||f||_\infty = \sup_{t \in T} |f(t)|.$$

10.11. Exercise. The trigonometric polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ are dense in $C(\mathbb{T})$, so $C(\mathbb{T})$ is separable.

The dual of $C(\mathbb{T})$ is identified by the Riesz Representation Theorem.

First recall that a *positive Borel measure* on $\mathbb{T}$ is a function μ : {Borel subsets of $\mathbb{T}$} $\to [0, \infty]$ such that (i) $\mu(\emptyset) = 0$ and (ii) $\mu(\bigcup_{i \in \mathbb{N}} A_i) = \sum \mu(A_i)$, if A_i are pairwise disjoint Borel sets. It is *finite,* if $\mu(\mathbb{T}) < \infty$, and a *probability* measure, if $\mu(\mathbb{T}) = 1$. A *complex Borel measure* is a map μ : {Borel subsets of $\mathbb{T}$} $\to \mathbb{C}$ which satisfies the above properties (i) and (ii). It turns out that every complex Borel measure μ can be written as $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\mu_1, \mu_2, \mu_3, \mu_4$ are *finite* positive Borel measures. It follows that there is $C < \infty$ such that

$$\sum_{i=1}^{\infty} |\mu(E_i)| \leq C$$

for any Borel partition $\{E_i\}_{i=1}^{\infty}$ of $\mathbb{T}$. Put

$$||\mu||_M = \sup \sum_{i=1}^{\infty} |\mu(E_i)|,$$

where the sup is over all these partitions. Notice that $||\mu||_M = \mu(\mathbb{T})$, if μ is a finite positive Borel measure.

Denote by $M(\mathbb{T})$ the vector space of complex Borel measures on $\mathbb{T}$ (where we put $(\alpha\mu + \lambda\nu)(E) = \alpha\mu(E) + \lambda\nu(E)$). Then $M(\mathbb{T})$ with the norm $||\mu||_M$ is a Banach space.

10.12. Exercise. For each $x \in \mathbb{T}$, let δ_x be the *Dirac measure* at x, i.e., $\delta_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$. Show that if $x \neq y$, $||\delta_x - \delta_y|| = 2$. Conclude that $M(\mathbb{T})$ is not separable.

The Riesz Representation Theorem identifies $C(\mathbb{T})^*$ with $M(\mathbb{T})$. Let's explain this more carefully. First, one can define for each $f \in C(\mathbb{T})$ the integral $\int f d\mu$ and show the properties

(i) $\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$,

(ii) $|\int f d\mu| \leq ||f||_\infty ||\mu||_M$,

so that μ gives rise to the element

$$f \mapsto \int f d\mu$$

of $C(\mathbb{T})^*$. The Riesz Representation Theorem asserts that these are all the elements of $C(\mathbb{T})^*$. More precisely, to each $\Lambda \in C(\mathbb{T})^*$ one can associate a unique complex Borel measure μ on $\mathbb{T}$ such that

$$\Lambda(f) = \int f d\mu, \text{ for } f \in C(\mathbb{T}).$$

Moreover $\Lambda \leftrightarrow \mu$ is a Banach space isomorphism between $C(\mathbb{T})^*$ and $M(\mathbb{T})$.

So we identify $C(\mathbb{T})^*$ and $M(\mathbb{T})$,

$$C(\mathbb{T})^* = M(\mathbb{T}).$$

Each $\mu \in M(\mathbb{T})$ operates on $f \in C(\mathbb{T})$ by

$$\langle f, \mu \rangle = \langle \mu, f \rangle = \int f d\mu.$$

Now denote by

$$P(\mathbb{T})$$

the set of all probability Borel measures on $\mathbb{T}$. Since

$$\mu \in P(\mathbb{T}) \Rightarrow ||\mu|| = \mu(T) = 1,$$

it follows that $P(\mathbb{T}) \subseteq B_1(M(\mathbb{T}))$.

Another part of the Riesz Representation Theorem asserts that in the correspondence $\Lambda \leftrightarrow \mu$, μ is a positive measure iff Λ is positive, i.e., $\Lambda(f) \geq 0$ for $f \geq 0$. Thus $\mu \in C(\mathbb{T})$ is positive, i.e., $\mu(E) \geq 0$ for Borel E, iff $\int f d\mu \geq 0$ for any $f \in C(\mathbb{T})$, with $f \geq 0$. (This can be also proved directly by approximation arguments.) Thus $P(\mathbb{T})$ consists exactly of all members of $B_1(M(\mathbb{T}))$ which satisfy

$$\int 1 d\mu = 1, \ \forall f \in C(\mathbb{T}) \ (f \geq 0 \Rightarrow \int f d\mu \geq 0).$$

It follows that $P(\mathbb{T})$ is a closed subset of $B_1(M(\mathbb{T}))$ when the latter is equipped with the weak*-topology, so it is compact metrizable in this topology. To summarize:

The space $P(\mathbb{T})$ of probability Borel measures on $\mathbb{T}$ with the weak*-topology, i.e., the smallest topology for which the maps

$$\mu \mapsto \int f d\mu, \ f \in C(\mathbb{T})$$

are continuous, is compact metrizable.

Notice again that the sets of the form

$$V_{f_1,\cdots,f_n,\epsilon} = \{\nu : \left| \int f_i d\mu - \int f_i d\nu \right| < \epsilon, \ i = 1, \cdots, n\},$$

where $\epsilon > 0$ and $f_1, \cdots, f_n \in C(\mathbb{T})$, form a nbhd basis of $\mu \in P(\mathbb{T})$ in the weak*-topology.

A *finite support probability measure* is a measure of the form $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$, where δ_x is the Dirac measure at x, and $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$. We say that μ is *supported* by $\{x_1, \cdots, x_n\}$.

10.13. Proposition. *Let $D \subseteq \mathbb{T}$ be a dense set in $\mathbb{T}$. Then the set of finite support probability measures supported by D is dense in $P(\mathbb{T})$ with the weak*-topology.*

Proof. Fix $\mu \in P(\mathbb{T})$ and an open nbhd $\{\nu : |\int f_i d\mu - \int f_i d\nu| < \epsilon, \ i = 1, \cdots, n\}$ $(f_1, \cdots, f_n \in C(\mathbb{T}), \epsilon > 0)$ of μ in the weak*-topology. We want to find a finite support probability measure ν supported by D which belongs in this nbhd. Since the functions $f_1, \cdots, f_n$ are uniformly continuous, we can find $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f_i(x) - f_i(y)| \leq \epsilon_1 < \epsilon,$$

for any $x, y \in \mathbb{T}$ and $i = 1, \cdots, n$. So there is a finite partition $I_1, \cdots, I_k$ of $\mathbb{T}$ into half-open intervals such that

$$x, y \in I_j \Rightarrow |f_i(x) - f_i(y)| \leq \epsilon_1, \ j \leq k, i \leq n.$$

Choose $x_i \in I_i \cap D$, let $\alpha_i = \mu(I_i)$ and put $\nu = \sum \alpha_i \delta_{x_i}$. Then ν is supported by D and for $f \in \{f_1, \cdots, f_n\}$

$$\left| \int f d\mu - \int f d\nu \right|$$

$$= \left| \int f d\mu - \sum \alpha_i f(x_i) \right| \text{ (since } \int f d(\delta_x) = f(x))$$

$$= \left| \sum_{i=1}^{k} (\int_{I_i} f d\mu - \int_{I_i} f(x_i) d\mu) \right| \text{ (since } \mu(I_i) = \alpha_i)$$

$$= \left| \sum_{i=1}^{k} \int_{I_i} (f(x) - f(x_i)) d\mu \right|$$

$$\leq \sum_{i=1}^{k} \int_{I_i} \left| f(x) - f(x_i) \right| d\mu$$

$$\leq \sum_{i=1}^{k} \int_{I_i} \epsilon_1 d\mu$$

$$= \epsilon_1 \int d\mu = \epsilon_1 < \epsilon.$$

$\dashv$

We can generalize this as follows. Fix a closed set $E \subseteq \mathbb{T}$ and let

$$P(E) = \{\mu \in P(\mathbb{T}) : \mu(E) = 1\}.$$

We claim that $P(E)$ is closed in the weak*-topology of $P(\mathbb{T})$, thus also compact. To see this, let $I_1, I_2, \cdots$ enumerate a sequence of open intervals whose union is the complement of E and note that $\mu \in P(E) \Leftrightarrow \forall n(\mu(I_n) = 0)$. If $C_n = \{f \in C(\mathbb{T})\colon f \text{ vanishes outside } I_n\}$, then clearly χ_{I_n} is the pointwise limit of a sequence $f_i \leq 1$ from C_n, so by the Lebesgue Dominated Convergence Theorem, $\mu(I_n) = 0 \Leftrightarrow \int \chi_{I_n} d\mu = 0 \Leftrightarrow \forall f \in C_n(\int f d\mu = 0)$, so $P(E)$ is an intersection of closed sets, thus is closed.

10.14. Exercise. Let $E \subseteq \mathbb{T}$ be closed and let $D \subseteq E$ be dense in E. Then the probability measures supported by finite subsets of D are dense in $P(E)$ with the weak*-topology.

There is a very interesting connection between $M(\mathbb{T})$ and ℓ^∞. To each $\mu \in M(\mathbb{T})$ associate its Fourier coefficients

$$\hat{\mu}(n) = \int e^{-int} d\mu.$$

As $|\hat{\mu}(n)| \leq |\int e^{-int} d\mu| \leq ||\mu||_M$, we see that $\hat{\mu} \in \ell^\infty$. Next note that if $\hat{\mu} = \hat{\nu}$ for two complex measures μ, ν, then $\int f d\mu = \int f d\nu$ for any trigonometric polynomial f and thus, as these are dense in $C(\mathbb{T})$, $\int f d\mu = \int f d\nu$ for $f \in C(\mathbb{T})$, i.e., $\mu = \nu$. So $\mu \mapsto \hat{\mu}$ is an injection from $M(\mathbb{T})$ into ℓ^∞. Denote by $\widehat{M(\mathbb{T})}$ its image, and more generally for $A \subseteq M(\mathbb{T})$ let $\hat{A} = \{\hat{\mu} : \mu \in A\}$. Then $\widehat{P(\mathbb{T})} \subseteq B_1(\ell^\infty)$. We claim that actually $\mu \mapsto \hat{\mu}$ is a homeomorphism of $P(\mathbb{T})$ with the weak*-topology and $\widehat{P(\mathbb{T})}$ with the weak*-topology (of $B_1(\ell^\infty)$, where $\ell^\infty = (\ell^1)^*$). Since $P(\mathbb{T})$ is compact, it is enough to show that $\mu \mapsto \hat{\mu}$ is continuous. So let $\mu_n \to \mu$ in the weak*-topology of $P(\mathbb{T})$. Then $\int f d\mu_n \to f d\mu$ for any $f \in C(\mathbb{T})$, so in particular $\hat{\mu}_n(i) \to \hat{\mu}(i)$, thus $\hat{\mu}_n \to \hat{\mu}$ in the weak*-topology of $\widehat{P(\mathbb{T})}$.

So we can identify, for all practical purposes, $P(\mathbb{T})$ and $\widehat{P(\mathbb{T})}$ by identifying μ with $\hat{\mu}$, so we often view $P(\mathbb{T})$ as a subset of ℓ^∞.

§11. Rajchman measures and the proof of the Debs-Saint Raymond Theorem.

We say that $\mu \in P(\mathbb{T})$ is a *Rajchman* measure if $\hat{\mu}(n) \to 0$, as $|n| \to \infty$. (By the way, Rajchman was Zygmund's teacher.) Denote their class by R. Then Theorem 9.2 says that for every non-meager $A \subseteq \mathbb{T}$ with the BP, $\exists \mu \in R(\mu(A) = 1)$.

We will now give the promised proof of 9.2, due to Kechris-Louveau, which is based on the Baire category method.

The streamlined presentation below is due to Lyons.

First, since $A \subseteq \mathbb{T}$ has the property of Baire, there is an open set $U \subseteq \mathbb{T}$, with $A \Delta U$ meager. As A is not meager, $U \neq \emptyset$, so there is a closed interval $I = [a, b], a \neq b$, and a sequence $U_n \subseteq I$ of open sets, dense in I, with $\bigcap_n U_n \subseteq A$.

Let for $E \subseteq \mathbb{T}$

$$R(E) = \{\mu \in R : \mu(E) = 1\}.$$

We want to show that $R(A) \neq \emptyset$. If we try to apply the category method in $P(\mathbb{T})$ with the weak*-topology, we run into a problem since $R(A) \subseteq R$ and R is unfortunately meager in $P(\mathbb{T})$ with the weak*-topology. (See 11.1 below). The trick is to work instead with $\hat{R} \subseteq c_0$ and the norm-topology.

Claim 1. $\hat{R}$ is a norm-closed subset of c_0. More generally, if $E \subseteq \mathbb{T}$ is closed, $\widehat{R(E)}$ is a norm-closed subset of c_0.

Proof. Take $\mu_n \in R(E)$ and assume $\hat{\mu}_n \to x \in c_0$ in norm, i.e., $||\hat{\mu}_n - x||_\infty \to 0$. Now this implies immediately that $\hat{\mu}_n(i) \to x(i)$, $\forall i \in \mathbb{Z}$. But recall that $P(\mathbb{T})$ is compact, so there is a subsequence $n_0 < n_1 < \cdots$ with $\mu_{n_j} \to \mu \in P(\mathbb{T})$ (for some $\mu \in P(\mathbb{T})$), with respect to the weak*-topology, so $\hat{\mu}_{n_j}(i) \to \hat{\mu}(i)$, $\forall i \in \mathbb{Z}$, i.e., $\hat{\mu}(i) = x(i)$, so $x = \hat{\mu} \in \hat{R}$. We now want to show that $\mu \in P(E)$, i.e., $\mu(E) = 1$, so that $\hat{\mu} \in \widehat{R(E)}$. But this is clear as $P(E)$ is closed in the weak*-topology of $P(\mathbb{T})$. ⊣

So $\widehat{R(I)}$ is in particular a complete metric space and we can apply the Baire Category Theorem to it. It will be clearly enough to show that each $\widehat{R(U_n)}$ is dense, G_δ in $\widehat{R(I)}$

in the norm-topology of $\widehat{R(I)}$. Because then $\bigcap_n \widehat{R(U_n)}$ is dense, G_δ in $\widehat{R(I)}$ and so non-empty, i.e., there is $\mu \in R$ such that for each $n, \mu(U_n) = 1$, thus $\mu(\bigcap_n U_n) = 1$ and since $\bigcap_n U_n \subseteq A,\ \mu(A) = 1$, and the proof is complete.

Thus our final claim is:

Claim 2. If I is a closed non-trivial interval in $\mathbb{T}$ and $U \subseteq I$ is open and dense in I, then $\widehat{R(U)}$ is dense G_δ in $\widehat{R(I)}$, in the norm-topology of $\widehat{R(I)}$.

Proof. First we check that $\widehat{R(U)}$ is G_δ in $\widehat{R(I)}$ in the norm-topology. We have

$$\widehat{R(U)} = \bigcap_{n\geq 1} \bigcup_{\substack{f\in C(\mathbb{T}) \\ 0\leq f\leq \chi_U}} \{\hat{\mu} \in \widehat{R(I)} : \int f d\mu > 1 - \frac{1}{n}\}.$$

(To see this recall that U is a disjoint union of open intervals.) It is then enough to check that for each $f \in C(\mathbb{T})$,

$$\{\hat{\mu} \in \widehat{R(I)} : \int f d\mu > 1 - \frac{1}{n}\}$$

is open in $\widehat{R(I)}$ in the norm-topology of c_0. In fact, we can easily see that it is open in $\widehat{R(I)}$ in the weak-topology of c_0. This is because

$$\mu \mapsto \int f d\mu$$

is continuous in the weak*-topology of $R(I)$ and thus in the weak*-topology of $\widehat{R(I)}$. But since $\widehat{R(I)} \subseteq c_0$, the weak*-topology of $\widehat{R(I)}$ is the same as the weak-topology of $\widehat{R(I)}$.

It remains to prove that $\widehat{R(U)}$ is dense in $\widehat{R(I)}$ for the norm-topology. Since clearly $\widehat{R(U)}$ is a convex subset of c_0, it is enough, by Mazur's Theorem 10.3, to show that $\widehat{R(U)}$ is weakly dense in $\widehat{R(I)}$. But again, as $\widehat{R(I)} \subseteq c_0$, this is the same thing as saying that $\widehat{R(U)}$ is weak*-dense in $\widehat{R(I)}$, where we now view these as subsets of ℓ^∞. But this again means the same thing as $R(U)$ being weak*-dense in $R(I)$, where we work in $P(\mathbb{T})$ now. We will in fact show that $\overline{R(U)}^{w^*}$ (= the weak*-closure of $R(U)$ in $P(\mathbb{T})$) $= P(I)$, which of course completes the proof. Since the probability measures with finite support contained in U are dense in $P(I)$, and $R(U)$ is convex, it is enough to show that every Dirac measure δ_x, with $x \in U$, is the limit of a sequence in $R(U)$ in the weak*-topology. But this is easy. Let $I_n \subseteq U$ be a decreasing sequence of open intervals with $\lambda(I_n) < \frac{1}{n}$ and $\{x\} = \bigcap_n I_n$. Let $\mu_n = (\lambda|I_n)/\lambda(I_n)$. Then, by direct calculation, $\hat{\mu}_n(i) \to 0$, as $|i| \to \infty$, so $\mu_n \in R(U)$. Now $\mu_n \to \delta_x$ in the weak*-topology. This is because for any $f \in C(\mathbb{T})$,

$$\int f d\mu_n - \int f d(\delta_x) = \frac{1}{\lambda(I_n)} \int_{I_n} f(t)dt - f(x)$$

$$= \frac{1}{\lambda(I_n)} \int_{I_n} (f(t) - f(x))dt,$$

so that if ϵ is given and n is large enough, so that $|f(t) - f(x)| < \epsilon$ for $t \in I_n$, we have

$$\left| \int f d\mu_n - \int f d(\delta_x) \right| < \epsilon.$$

This completes the proof.

11.1. Exercise. Show that R is meager in $P(\mathbb{T})$ with the weak*-topology.

Although R is meager, it still has an interesting largeness property. Note that for a set A in a complete metric space, if A has the BP, then A is comeager iff for each open non-$\emptyset$ set $U \subseteq X$ and each sequence of open dense in U sets, U_n, we have $A \cap (\bigcap_n) U_n \neq \emptyset$. Now R, although meager, is *convexly comeager,* in the following sense.

11.2. Theorem (Kechris-Louveau). *For every non-empty open set $U \subseteq P(\mathbb{T})$ (in the weak*-topology) and any sequence U_n of open dense in U* **convex** *sets, we have $R \cap (\bigcap_n U_n) \neq \emptyset$.*

I will omit the proof (see Kechris-Louveau [1989], VIII. 3.6]).

11.3. Exercise. Use this to give another proof of the Debs-Saint Raymond Theorem.

§12. Paying a debt: Proof of 7.6.

To bring this chapter into conclusion I will give the proof that (i) $\Rightarrow$ (ii) in 7.6, which we omitted earlier. The proof will be based on a classical result of Riemann, known as the *localization principle.*

12.1. Riemann localization principle. *Let $S \sim \sum c_n e^{inx}$ be a trigonometric series with $c_n \to 0$. If the Riemann function F_S is linear in some open interval (a, b), then $\sum c_n e^{inx} = 0$ on (a, b) (and uniformly on closed subintervals).*

Note that the hypothesis is equivalent (by Schwartz's Lemma 3.3) to saying that $D^2 F_S(x) = 0$, $\forall x \in (a, b)$. Recall also Riemann's First Lemma 2.2 which implies that if $\sum c_n e^{inx} = 0$, then $D^2 F_S(x) = 0$. So 12.1 asserts a converse, but only under the hypothesis that $D^2 F_S(x)$ vanishes in a whole interval.

Before proving 12.1, let me first show how it can be used to prove (i) $\Rightarrow$ (ii) in 7.6. To start with, note the following fact.

12.2. Exercise. Let $f \in C(\mathbb{T})$ be such that $\sum |\hat{f}(n)| < \infty$. Then $f(x) = \sum \hat{f}(n) e^{inx}$ uniformly on x.

Now fix $a \in \mathbb{R}$, $0 < h < \pi$ and let $\psi_{a,h}$ be the 2π-periodic function defined in the period $[a - \pi, a + \pi]$ as follows: $\psi_{a,h}(a) = 2\pi/h, \psi_{a,h}(x) = 0$ off $[a - h, a + h]$, and $\psi_{a,h}$ in linear in $[a - h, a], [a, a + h]$. Then one can easily check that the Fourier series of $\psi_{a,h}$ is

$$S(\psi_{a,h}) \sim \sum_{n=-\infty}^{\infty} e^{-ina} \left(\frac{\sin(nh/2)}{nh/2} \right)^2 e^{inx},$$

so that $\sum |\hat{\psi}_{a,h}(n)| < \infty$ and

$$\psi_{a,h}(x) = \sum_{n=-\infty}^{\infty} e^{-ina} \left(\frac{\sin(nh/2)}{nh/2} \right)^2 e^{inx}.$$

Let us next note another fact.

12.3. Exercise. Let $f \in C(\mathbb{T})$ be such that $\sum |\hat{f}(n)| < \infty$. Then for any $\mu \in M(\mathbb{T})$,

$$\int f d\mu = \sum \hat{\mu}(n) \hat{f}(-n).$$

So if $E \subseteq \mathbb{T}$ is a closed set and $\mu(E) = 1$, we have for any $a \notin E$ and h small enough that

$$\begin{aligned} 0 = \int \psi_{a,h} d\mu &= \sum_{n=-\infty}^{\infty} \hat{\mu}(n) \hat{\psi}_{a,h}(-n) \\ &= \sum_{n=-\infty}^{\infty} \hat{\mu}(n) \left(\frac{\sin(nh/2)}{nh/2} \right)^2 e^{ina} \\ &= \frac{\Delta^2 F_S(a,h)}{h^2} \end{aligned}$$

where $S \sim \sum \hat{\mu}(n) e^{inx}$. So F_S is linear on each open interval disjoint from E and thus, by the Riemann Localization Principle, $\sum \hat{\mu}(n) e^{inx} = 0$ off E.

So it only remains to prove 12.1.

§13. The Rajchman Multiplication Theory.

We will first develop a theory, due to Rajchman, concerning the formal multiplication of trigonometric series by "nice" functions. Beyond being useful in proving the Riemann localization principle, it has many other applications, some of which we will see later on.

Let $S \sim \sum c_n e^{inx}$ have bounded coefficients $|c_n| \leq M < \infty$. Let $f \in C(\mathbb{T})$ have absolutely convergent Fourier coefficients $\sum |\hat{f}(n)| < \infty$, so that $f(x) = \sum \hat{f}(n) e^{inx}$ uniformly. Define the *formal product* $S(f) \cdot S$ (another trigonometric series) by

$$S(f) \cdot S \sim \sum C_n e^{inx},$$

where $C_n = \sum_k c_k \hat{f}(n-k)$. Clearly $\sum_k c_k \hat{f}(n-k)$ is convergent and $|C_n| \leq \sup_k |c_k| \cdot \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$.

13.1. Exercise. $c_n \to 0 \Rightarrow C_n \to 0$.

13.2. Lemma. *If* $\sum_{\ell=0}^{\infty} \sum_{|n|\geq \ell} |\hat{f}(n)| < \infty$ *(e.g., if* $\hat{f}(n) = O(\frac{1}{|n|^3})$*) and* $c_n \to 0$*, then* $\sum_{-N}^{N} C_n e^{inx} - f(x) \sum_{-N}^{N} c_n e^{inx} \to 0$ *uniformly on* x *(i.e.,* $\sum_{-\infty}^{\infty} (C_n e^{inx} - f(x) c_n e^{inx}) = 0$*, uniformly on* x*).*

Proof. First we prove that if $f(x) = 0$ for $x \in P \subseteq \mathbb{T}$, then $\sum_{-N}^{N} C_n e^{inx} = 0$ uniformly on $x \in P$. To see this note that

$$\begin{aligned}
\sum_{-N}^{N} C_n e^{inx} &= \sum_{n=-N}^{N} \Big(\sum_{k=-\infty}^{\infty} c_k e^{ikx} \hat{f}(n-k) e^{i(n-k)x} \Big) \\
&= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \left(\sum_{n=-N}^{N} \hat{f}(n-k) e^{i(n-k)x} \right) \\
&= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \left(\sum_{m=-N-k}^{N-k} \hat{f}(m) e^{imx} \right) \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \sum_{|k|\leq \frac{1}{2}N} \cdots,$$

$$I_2 = \sum_{|k|> \frac{1}{2}N} \cdots.$$

Since $\sum_{m=-\infty}^{\infty} \hat{f}(m) e^{imx} = 0$, $\sum_{m=-N-k}^{N-k} \hat{f}(m) e^{imx} = -\Big(\sum_{m=-\infty}^{-N-k-1} \hat{f}(m) e^{imx} + \sum_{m=N-k+1}^{\infty} \hat{f}(m) e^{imx}\Big)$, so

$$|I_1| \leq \sup_k |c_k| \cdot 2\Big(\sum_{\ell \geq N/2} \sum_{|m|\geq \ell} |\hat{f}(m)|\Big) \to 0$$

as $N \to \infty$, uniformly on x, since $\sum\limits_{\ell=0}^{\infty} \sum\limits_{|m|\geq \ell} |\hat{f}(m)| < \infty$. Also

$$|I_2| \leq \sup_{|k|>\frac{N}{2}} |c_k| \cdot \sum_{k\in\mathbb{Z}} \Big(\sum_{m=-\infty}^{-N-k-1} |\hat{f}(m)| + \sum_{m=-\infty}^{N-k+1} |\hat{f}(m)| \Big)$$

$$\leq \sup_{|k|>\frac{N}{2}} |c_k| \cdot 3 \Big(\sum_{\ell=0}^{\infty} \sum_{|m|\geq \ell} |\hat{f}|(m)| \Big) \to 0$$

uniformly on x (consider cases as $k \leq -N, k \in (-N, N), k \geq N$.)

Note that this argument applies as well to any series of the form

$$\sum \tilde{f}(n,x) e^{inx}$$

provided that $\sum_{|n|\geq \ell} |\tilde{f}(n,x)| \leq M_\ell$, with $\sum_{\ell \geq 0} M_\ell < \infty$. In this case $C_n = \sum\limits_k c_k \tilde{f}(n-k,x)$ and the hypothesis $f(x) = 0$ is replaced by $\sum \tilde{f}(n,x)e^{inx} = 0$.

Now consider the general situation. We have

$$\sum_{-N}^{N} C_n e^{inx} - f(x) \sum_{-N}^{N} c_n e^{inx}$$

$$= \sum_{-N}^{N} (C_n - f(x) c_n) e^{inx}.$$

But

$$C_n - f(x) c_n = \sum_{-\infty}^{\infty} c_k \hat{f}(n-k) - f(x) c_n$$

$$= \sum_{-\infty}^{\infty} c_k \tilde{f}(n-k,x),$$

where $\tilde{f}(m,x) = \begin{cases} \hat{f}(m), & \text{if } m \neq 0 \\ \hat{f}(0) - f(x), & \text{if } m = 0 \end{cases}$. Now $\sum \tilde{f}(n,x)e^{inx} = \sum \hat{f}(n)e^{inx} - f(x) = 0$ for all x. Also if $M_k = \sum\limits_{|n|\geq k} |\hat{f}(n)|$, for $k > 0$, and $M_0 = \sum |\hat{f}(n)| + ||f||_\infty$, we have $\sum\limits_{|n|\geq k} |\tilde{f}(n,x)| = M_k$ if $k > 0$ and $\sum\limits_n |\tilde{f}(n,x)| = M_0$. Since $\sum\limits_{k\geq 0} M_k < \infty$, it follows from the preceding remarks that

$$\sum_{-N}^{N} (C_n - f(x) c_n) e^{inx} \to 0$$

uniformly on x, i.e.,

$$\sum_{-N}^{N} C_n e^{inx} - f(x) \sum_{-N}^{N} c_n e^{inx} \to 0$$

uniformly on x. ⊣

We are now ready to prove the Riemann Localization Principle.

Let $S \sim \sum c_n e^{inx}$ be a trigonometric series with $c_n \to 0$ and assume F_S is linear on (a, b). Let $[c, d] \subseteq (a, b)$. Take a nice function $f \in C(\mathbb{T})$, say with continuous derivatives of all orders, such that $f = 1$ on $[c, d]$ and 0 off (a, b). Integration by parts shows that $\hat{f}(n) = O\left(\frac{1}{|n|^k}\right)$ for all $k \geq 0$, so in particular $\sum\limits_k \sum\limits_{|n| \geq k} |\hat{f}(n)| < \infty$. So

$$\sum_{-\infty}^{\infty} \left(C_n e^{inx} - f(x) c_n e^{inx} \right) = 0,$$

uniformly on x. Then by 2.3

$$\lim_{h \to 0} \sum_{-\infty}^{\infty} \left(C_n e^{inx} - f(x) c_n e^{inx} \right) \left(\frac{\sin nh}{nh} \right)^2 = 0.$$

Now F_S is linear on (a, b), so for $x \in (a, b)$ and small enough h,

$$\sum_{-\infty}^{\infty} c_n e^{inx} \left(\frac{\sin nh}{nh} \right)^2 = \frac{\Delta^2 F_S(x, 2h)}{4h^2} = 0,$$

so for $x \in (a, b)$

$$\lim_{h \to 0} \sum_{-\infty}^{\infty} f(x) c_n e^{inx} \left(\frac{\sin nh}{nh} \right)^2 = 0,$$

thus this is true for *all* x since $f(x) = 0$ off (a, b). So

$$\lim_{h \to 0} \sum_{-\infty}^{\infty} C_n e^{inx} \left(\frac{\sin nh}{nh} \right)^2 = 0.$$

But if $T \sim \sum C_n e^{inx}$, then this limit is simply $D^2 F_T(x)$, so

$$D^2 F_T(x) = 0$$

for all x, thus F_T is linear, and so $C_n = 0$ for all n, thus

$$\sum_{-\infty}^{\infty} f(x) c_n e^{inx} = 0$$

uniformly for all x, thus as $f(x) = 1$ on $[c, d]$, $\sum c_n e^{inx} = 0$ uniformly for $x \in [c, d]$. ⊣

PART III. DESCRIPTIVE METHODS

§14. Perfect sets of uniqueness.

Until now the only examples of sets of uniqueness that we have seen are the countable ones. So it is conceivable that $\mathcal{U}$ = countable. This turned out to be false since in the period 1921–23 Rajchman and Bari came up independently with examples of perfect sets of uniqueness. We will give here Rajchman's approach which makes use of his multiplication theory.

For $x \in [0, 2\pi]$ and $m \in \mathbb{Z}$ we let $mx = mx(\text{mod } 2\pi)$. (If we identify x with $e^{ix} \in \mathbb{T}$, then $mx = e^{imx}$.) For $A \subseteq \mathbb{T}$ we let $mA = \{mx : x \in A\}$. The next definition is due to Rajchman.

Definition. (Rajchman). A set $E \subseteq \mathbb{T}$ is called an *H-set* if for some nonempty open interval $I \subseteq \mathbb{T}$ and some sequence $0 \leq n_0 < n_1 < n_2 < \cdots$, we have $(n_k E) \cap I = \emptyset$ for all k.

Examples. (i) Every finite set is an H-set (but not every countable set).

(ii) The Cantor 1/3-set in $[0, 2\pi]$, i.e., the set E of numbers of the form $2\pi \sum_{n=1}^{\infty} \epsilon_n/3^n$, with $\epsilon_n = 0, 2$, is an H-set. Indeed, $3^n E$ avoids the middle 1/3 interval.

14.1. Theorem (Rajchman). *Every H-set is a set of uniqueness. So the Cantor* 1/3-*set is a set of uniqueness.*

Proof. Notice that the closure of an H-set is an H-set, so we will work with a closed H-set E. Let $I \neq \emptyset$ be an open interval and let $0 \leq n_0 < n_1 < \cdots$ be such that $(n_k E) \cap I = \emptyset$. Let $S \sim \sum c_n e^{inx}$ be a trigonometric series with $\sum c_n e^{inx} = 0$ off E. We will show that $c_n = 0$. Clearly $c_n \to 0$, by the Cantor-Lebesgue Lemma.

Choose a $f \in C(\mathbb{T})$ which has derivatives of all orders, $\hat{f}(0) = 1$ and $\text{supp}(f) = \overline{\{x : f(x) \neq 0\}} \subseteq I$. Put

$$f_k(x) = f(n_k x).$$

Then $f_k = 0$ on E. Let

$$S(f_k) \cdot S \sim \sum C_n^k e^{inx}.$$

Claim. $C_n^k \to c_n$, as $k \to \infty$

Since by 13.2 we have that

$$\sum_{-N}^{N} C_n^k e^{inx} - f_k(x) \sum_{-N}^{N} c_n e^{inx} \to 0$$

for all x, it follows that

$$\sum_{-\infty}^{\infty} C_n^k e^{inx} = 0$$

for all x (as $f_k(x) = 0$ on E and $\sum\limits_{-\infty}^{\infty} c_n e^{inx} = 0$ off E). So $C_n^k = 0$ and thus by the claim, $c_n = 0$.

Proof of the claim. Note that since

$$f(x) = \sum \hat{f}(n) e^{inx},$$

we have that

$$f_k(x) = f(n_k x) = \sum \hat{f}(n) e^{in \cdot n_k x},$$

thus

$$\hat{f}_k(i) = \begin{cases} \hat{f}(n), & \text{if } i = n \cdot n_k \\ \\ 0, & \text{otherwise} \end{cases},$$

so that $\sum\limits_{i \in \mathbb{Z}} |\hat{f}_k(i)| \le C < \infty$ for all k, $\hat{f}_k(0) = 1$, and $\lim\limits_{k \to \infty} \hat{f}_k(i) = 0$, for $i \neq 0$.

Now we have

$$\begin{aligned} C_n^k &= \sum_m c_{n-m} \hat{f}_k(m) \\ &= \sum_{|m| \le N} \cdots + \sum_{|m| > N} \cdots \end{aligned}$$

for any $N > |n|$. The first sum converges to c_n as $k \to \infty$ and the second is bounded by $\sup\{|c_k| : |k| \ge N - |n|\} \cdot C$, which goes to 0 as $N \to \infty$, so $C_n^k \to c_n$ as $k \to \infty$. ⊣

§15. The Characterization Problem and the Salem-Zygmund Theorem.

We have now seen that (for measurable sets)

$$\text{countable} \subsetneqq \mathcal{U} \subsetneqq \text{null},$$

so an attempt to identify the sets of uniqueness with other types of "thin" sets like countable or null has failed. This raises the more general question of whether it is possible to characterize in some sense the sets of uniqueness. This problem was already prominent in the 1920's and in fact even in the simplest case, that of closed sets or even perfect sets. For example, in Bari's memoir on the problems of uniqueness in Fundamenta Mathematicae, Bari [1927], the following problem is explicitly stated.

The Characterization Problem. *Find a necessary and sufficient condition for a perfect set to be a set of uniqueness.*

As usual with such characterization problems this is a somewhat vague question. It appears though that the intended meaning was to find a characterization which can be expressed fairly explicitly in terms of a standard description of a given perfect set E, e.g., its sequence of contiguous intervals. Many attempts have been made to obtain such a characterization (see, e.g., Bari's monograph, Bari [1964]) without success in the general case of an arbitrary perfect set. However, in the 1950's Salem and Zygmund, following earlier work of Bari and Piatetski-Shapiro, proved a remarkable theorem which characterized when a perfect symmetric set of constant ratio of dissection is a set of uniqueness. We will next state the Salem-Zygmund Theorem.

Fix a sequence of numbers $\xi_1, \xi_2, \cdots$ with $0 < \xi_i < 1/2$. The *symmetric perfect set with dissection ratios* $\xi_1, \xi_2, \cdots$, in symbols $E_{\xi_1,\xi_2,\ldots}$, is defined as follows: For each interval $[a, b]$, and $0 < \xi < 1/2$ consider the middle open interval $(a + \xi\ell, b - \xi\ell)$, where $\ell = b - a$, and let $E = [0, a + \xi\ell] \cup [b - \xi\ell, b]$ be the remaining closed intervals. We say that E is obtained from $[a, b]$ by a dissection of ratio ξ. Starting with $[0, 2\pi]$ define $E_1 \supseteq E_2 \supseteq \cdots$, where E_k is a union of 2^k closed intervals in $[0, 2\pi]$, by letting E_1 be obtained from $[0, 2\pi]$ by a dissection of ratio ξ_1 and E_{k+1} be obtained form E_k by applying a dissection of ratio ξ_{k+1} to each interval of E_k. Let

$$E_{\xi_1,\xi_2,\cdots} = \bigcap_k E_k.$$

Then $E_{\xi_1,\xi_2,\cdots}$ is a perfect nowhere dense set and $\lambda(E_{\xi_1,\xi_2,\ldots}) = 0$ iff $2^k \xi_1 \xi_2 \cdots \xi_k \to 0$.

If $\xi = \xi_1 = \xi_2 = \cdots$, we write E_ξ instead of $E_{\xi_1,\xi_2,\cdots}$ and call E_ξ the *symmetric perfect set of constant ratio of dissection* ξ. Clearly (as $\xi < 1/2$) $\lambda(E_\xi) = 0$. The classical Cantor set is the set $E_{1/3}$.

The Salem-Zygmund Theorem characterizes when E_ξ is a set of uniqueness. Remarkably this depends on a subtle number theoretic property of ξ. We need the following definition.

Definition. An algebraic number θ is called an *algebraic integer* if θ is the root of a polynomial $P(x) \in \mathbb{Z}[x]$ with leading coefficient 1. Then there is a unique polynomial $P(x)$ of least degree with leading coefficient 1 for which $P(\theta) = 0$, called the *minimal polynomial* by θ. Say it has degree $n \geq 1$. Write $\theta = \theta^{(1)}, \theta^{(2)}, \cdots, \theta^{(n)}$ for its roots. We call $\theta^{(2)}, \cdots, \theta^{(n)}$ the *conjugates* of θ. We say that θ is a *Pisot number* if $\theta > 1$ and all its conjugates have absolute value < 1. (So θ must be real.)

Examples. (i) Every $n \in \mathbb{N}, n \geq 2$ is Pisot, since it satisfies $x - n = 0$ which has only one root.

(ii) $\frac{1+\sqrt{5}}{2}$ is Pisot, since it satisfies $x^2 - x - 1 = 0$ and its conjugate is $\frac{1-\sqrt{5}}{2}$ with $|\frac{1-\sqrt{5}}{2}| < 1$.

(iii) A rational p/q is Pisot iff it is an integer > 1 (otherwise it is not an algebraic integer).

(iv) $\sqrt{2}$ is not Pisot.

Intuitively, a Pisot number is a number $\theta > 1$, whose powers θ^m are "almost" integers. To see this first let $\theta^{(2)}, \cdots \theta^{(n)}$ be the conjugates of the Pisot number θ. Then $\theta^m + (\theta^{(2)})^m + \cdots + (\theta^{(n)})^m$ is a symmetric polynomial of the roots of the minimal polynomial

$P(x)$ of θ, so it is an integer. But $|\theta^{(i)}| < 1$, so $(\theta^{(i)})^m \to 0$ as $m \to \infty$, for $i > 1$, so $\theta^{(m)}$ is closer and closer to an integer as $m \to \infty$. Conversely, it can be shown that if we let $\{x\}$ = distance of x to the nearest integer, and $\theta > 1$ is such that $\sum\limits_{n=0}^{\infty} \{\theta^n\}^2 < \infty$, then θ is a Pisot number, and if θ is already algebraic, then $\{\theta^n\} \to 0$ is enough.

One remarkable fact about Pisot numbers is the following:

(Pisot) *The set of Pisot numbers is closed (and of course countable).*

It turns out that it has Cantor-Bendixson rank exactly ω.

We now have:

15.1. The Salem-Zygmund Theorem. *Let $0 < \xi < \frac{1}{2}$ and let E_ξ be the symmetric perfect set of constant ratio dissection ξ Then*

$$E_\xi \textit{ is a set of uniqueness } \Leftrightarrow \theta = \frac{1}{\xi} \textit{ is Pisot.}$$

Thus it appears that number theoretic issues enter into the arena of the characterization problem.

Salem and Zygmund extended somewhat their theorem to a wider class of perfect sets. We will state this generalization for further reference.

Fix $\eta_0 = 0 < \eta_1 < \cdots < \eta_k < \eta_{k+1} = 1$ and put $\xi = 1 - \eta_k$. Assume that $\xi < \eta_{i+1} - \eta_i$ for $i < k$. The so-called *homogeneous perfect set associated to* $(\xi; \eta_1, \cdots, \eta_k)$, $E(\xi; \eta_1, \cdots \eta_k)$ is defined as follows:

For each closed interval $[a, b]$ with length $\ell = [a, b]$, consider the disjoint intervals $[a + \ell\eta_i, a + \ell\eta_i + \ell\xi], i = 0, \cdots, k$ and let E be their union. We say then that E results from $[a, b]$ by a dissection of type $(\xi; \eta_1, \cdots, \eta_k)$. Starting from $[0, 2\pi]$ define closed sets of $E_1 \supseteq E_2 \supseteq \cdots$ by performing to each interval of E_n a dissection of type $(\xi; \eta_1, \cdots, \eta_k)$ to get E_{n+1}, and let

$$E(\xi; \eta_1, \cdots, \eta_k) = \bigcap_n E_n.$$

Clearly E_n is made up of $(k+1)^n$ intervals of length ξ^n, so, as $(k+1)\xi < 1$ we have that $(k+1)^n \xi^n \to 0$, thus $\lambda(E(\xi; \eta_1 \cdots \eta_k)) = 0$. Note that $E(\xi; \eta_1) = E(\xi; 1 - \xi) = E_\xi$.

We now have:

15.2. The General Salem-Zygmund Theorem. *The set $E(\xi; \eta_1, \cdots, \eta_k)$ is a set of uniqueness iff*

$$\text{(i) } \theta = \frac{1}{\xi} \text{ is Pisot}$$

and

$$\text{(ii) } \eta_1, \cdots, \eta_k \in \mathbb{Q}(\theta).$$

The proof of 15.2 can be found in Kahane-Salem [1994].

This is essentially the best known positive result concerning the Characterization Problem. For example, there is no known characterization of when $E_{\xi_1,\xi_2,\ldots}$ is a set of uniqueness. Any such potential characterization would have to look quite different since Meyer has shown that if $\sum \xi_n^2 < \infty$, then $E_{\xi_1,\xi_2,\ldots}$ is a set of uniqueness.

In the preface of Zygmund's classic treatise, Zygmund [1979], he states: "*Two other major problems of the theory also await their solution. These are the structure of the sets of uniqueness and the structure of the functions with absolutely convergent Fourier series .... in a search for solutions we shall probably have to go beyond the domains of the theory of functions, in the direction of the theory of numbers and Diophantine approximation.*" (This was of course written after the proof of the Salem-Zygmund Theorem, which was proved in 1955.)

In the rest of this chapter we will develop another approach to the Characterization Problem based on the concepts and methods of descriptive set theory. This approach has led also to other significant dividends, as, for example, the original solution of the Category Problem.

This approach, based on the idea of studying the global structure of the class of closed sets of uniqueness from a descriptive standpoint, has led to interesting conclusions concerning the Characterization Problem for arbitrary perfect sets, by providing sharp limitations on the possibility of a positive solution. Whether these results actually provide a negative solution to the Characterization Problem is a matter of interpretation of the original question, which is rather vague. It certainly rules out characterizations of the type that researchers in the field have tried to establish over the years. Independently, of this, the point of view and the techniques that will be explained in the sequel should be useful in general in attacking similar characterization problems in analysis or other areas of mathematics.

§16. The hyperspace $\mathbf{K}(\mathbb{T})$ of closed subsets of the circle.

Descriptive set theory is the study of "definable" sets in *Polish,* i.e., complete separable metric spaces (like $\mathbb{R}, \mathbb{C}, \mathbb{R}^n$, etc.). In this theory, sets in such spaces are classified in hierarchies according to the complexity of their definitions, and the structure of the sets in each level of these hierarchies is studied in detail.

We want to apply this theory to the study of the global structure of *closed* sets of uniqueness which we will denote by U:

$$U = \{E \subseteq \mathbb{T} : E \text{ is a closed set of uniqueness}\}.$$

It is also important to consider a wider class, the so-called sets of *extended uniqueness* which are those sets $E \subseteq \mathbb{T}$ which satisfy uniqueness for series of the form $\sum \hat{\mu}(n)e^{inx}$ with $\mu \in M(\mathbb{T})$. We denote the class by $\mathcal{U}_0$, so that

$$E \in \mathcal{U}_0 \Leftrightarrow \text{ for every } \mu \in M(\mathbb{T}), \text{ if } \sum \hat{\mu}(n)e^{inx} = 0 \text{ off } E, \text{ then } \hat{\mu}(n) = 0, \forall n \in \mathbb{Z}.$$

We also let

$$U_0 = \{E \subseteq \mathbb{T} : E \text{ is a closed set of extended uniqueness}\}.$$

Clearly $U \subseteq U_0$ (and Piatetski-Shapiro actually showed that this inclusion is proper).

Both the sets U, U_0 are "definable" subsets (and we will see later on at what level of complexity) of the so-called *hyperspace* of $\mathbb{T}$, i.e., the space of all closed subsets of the circle equipped with an appropriate topology, which we will now describe.

Let us start more generally with a compact metric space (X, d), with $d \leq 1$ (for normalization purposes), like, e.g., the circle $\mathbb{T}$ with the usual metric (normalized arc-length). Denote by $K(X)$ the set of all closed (= compact) subsets of X. Define on $K(X)$ the following metric, called the *Hausdorff metric* (associated to d), d_H:

$$\begin{aligned} d_H(K, L) &= 0, \text{ if } K = L = \emptyset, \\ &= 1, \text{ if } K \neq L, K = \emptyset \text{ or } L = \emptyset, \\ &= \max\{\delta(K, L), \delta(L, K) : K, L \neq \emptyset\}, \end{aligned}$$

where

$$\delta(K, L) = \max_{x \in K} d(x, L).$$

16.1. Exercise. If

$$B(E, \epsilon) = \{x : d(x, E) < \epsilon\},$$

show that

$$d_H(K, L) < \epsilon \Leftrightarrow K \subseteq B(L, \epsilon) \;\&\; L \subseteq B(K, \epsilon).$$

16.2. Exercise. (i) Show that $(K(X), d_H)$ is complete. *Hint.* If $\{K_n\}$ is Cauchy with $K_n \neq \emptyset$, then $K = \bigcap_n (\overline{\bigcup_{i=n}^{\infty} K_i})$ is the limit of $\{K_n\}$.

(ii) Show that $(K(X), d_H)$ is compact. *Hint.* If $F \subseteq X$ is finite, with $\forall x \in X \exists y \in F(d(x, y) < \epsilon)$, then $K(F) = \{K \in K(\mathbb{T}) : K \subseteq F\}$ is finite and $\forall K \in K(X) \exists L \in K(F)(d_H(K, L) < 2\epsilon)$.

(iii) Show that if $D \subseteq X$ is dense, then $K_f(D) = \{K \in K(\mathbb{T}) : K \subseteq D, K \text{ finite}\}$ is dense in $K(X)$.

Thus $(K(X), d_H) \equiv K(X)$ is a compact metric space, so it is separable, and thus a Polish space.

Although the metric on $K(X)$ depends on the chosen metric on X, the topology of $K(X)$ depends only on the topology of X.

For any topological space X, we let $K(X)$ be the space of compact subsets of X. We give $K(X)$ the so-called *Vietoris topology* which is the one generated by the sets

$$\{K \in K(X) : K \subseteq U\},$$

$$\{K \in K(X) : K \cap U \neq \emptyset\},$$

where $U \subseteq X$ is open. So a basis of this topology is given by the sets

$$\{K \in K(X) : K \subseteq U_0 \,\&\, K \cap U_1 \neq \emptyset \,\&\, \cdots \,\&\, K \cap U_n \neq \emptyset\}$$

for $U_0, U_1, \cdots, U_n \subseteq X$ open.

16.3. Exercise. Show that the topology of $(K(X), d_H)$ is exactly the Vietoris topology on $K(X)$.

The following facts are not hard to prove. (Sometimes the best method is to use 16.3.)

16.4. Exercise. (i) $x \mapsto \{x\}$ is an isometry of X into $K(X)$.

(ii) $\{(x, K) : x \in K\}$, $\{(K, L) : K \subseteq L\}$, $\{(K, L) : K \cap L \neq \emptyset\}$, are closed in $X \times K(X), K(X) \times K(X)$, resp.

(iii) $(K, L) \mapsto K \cup L$ is continuous (from $K(X) \times K(X)$ into $K(X)$) but, in general $(K, L) \mapsto K \cap L$ is not.

(iv) If $f : X \to Y$ is continuous, so is $f'' : K(X) \to K(Y)$ given by $f''(K) = f[K]$.

(v) The operation $\bigcup : K(K(X)) \to K(X)$ given by $\bigcup \mathcal{K} = \bigcup\{K : K \in \mathcal{K}\}$ for any closed $\mathcal{K} \subseteq K(X)$ is continuous.

(vi) $K_f(X) = \{K \in K(X) : K \text{ is finite}\}$ is F_σ in $K(X), K_p(X) = \{K \in K(X) : K$ is perfect$\}$ is G_δ in $K(X)$.

(vii) If $A \subseteq X$, let

$$K(A) = \{K \in K(X) : K \subseteq A\}.$$

If A is closed, open, G_δ, then $K(A)$ is closed, open, G_δ, resp.

§17. Review of descriptive set theory.

Our reference for the concepts and results of descriptive set theory that we will use here is Kechris [1995].

Let X be a Polish space. A set $A \subseteq X$ is *Borel* if it belongs to the smallest σ-algebra containing the open sets. So all open, closed, $F_\sigma, G_\delta, \cdots$ sets are Borel. We ramify Borel sets in a transfinite hierarchy of ω_1 (= the first uncountable ordinal) stages, called the Borel hierarchy. We let

$$\mathbf{\Sigma}^0_1 = \text{ open}, \ \mathbf{\Pi}^0_1 = \text{ closed},$$

and for $\alpha < \omega_1$, we inductively define $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha$ by

$$\mathbf{\Sigma}^0_\alpha = \{\bigcup_n A_n : A_n \text{ is } \mathbf{\Pi}^0_{\alpha_n} \text{ for some } \alpha_n < \alpha\},$$

$$\mathbf{\Pi}^0_\alpha = \{X \setminus A : A \in \mathbf{\Sigma}^0_\alpha\}.$$

So $\mathbf{\Sigma}^0_2 = F_\sigma, \mathbf{\Pi}^0_2 = G_\delta, \mathbf{\Sigma}^0_3 =$ countable unions of G_δ sets $= G_{\delta\sigma}, \mathbf{\Pi}^0_3 =$ complements of $\mathbf{\Sigma}^0_3$ sets $=$ countable intersections of F_σ sets $= F_{\sigma\delta}$, etc.

Let also

$$\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha,$$

so that $\mathbf{\Delta}^0_1 =$ clopen, for example. To emphasize that we work in the space X, we denote this also by $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ if necessary. We also let $\mathbf{B} = \mathbf{B}(X)$ be the class of Borel sets in X.

We have $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_{\alpha+1} \cap \mathbf{\Pi}^0_{\alpha+1} = \mathbf{\Delta}^0_{\alpha+1}$, so that we have an increasing hierarchy of sets and

$$\mathbf{B} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha \Big(= \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Delta}^0_\alpha \Big).$$

We call $\{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha\}_{\alpha<\omega_1}$ the *Borel hierarchy.* It is proper, i.e., $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subsetneqq \mathbf{\Delta}^0_{\alpha+1}$, if X is uncountable.

A subset $A \subseteq X$ is $\mathbf{\Sigma}^1_1$ or *analytic* if for some Polish space Y, Borel $B \subseteq Y$, and continuous $f : Y \to X$ we have $f[B] = A$. A set $A \subseteq X$ is $\mathbf{\Pi}^1_1$ or *co-analytic* if $X \setminus A$ is $\mathbf{\Sigma}^1_1$. Inductively define

$$\mathbf{\Sigma}^1_{n+1} = \text{ the class of continuous images of } \mathbf{\Pi}^1_n \text{ sets,}$$

$$\mathbf{\Pi}^1_{n+1} = \text{ the complements of } \mathbf{\Sigma}^1_{n+1} \text{ sets.}$$

Also put $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$. Again we write $\mathbf{\Sigma}^1_n(X), \mathbf{\Pi}^1_n(X), \mathbf{\Delta}^1_n(X)$ to emphasize that we look at subsets of X. It turns out that $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$. The *projective* subsets of X are defined by

$$\mathbf{P} = \mathbf{P}(X) = \bigcup_n \mathbf{\Sigma}^1_n(X) (= \bigcup_n \mathbf{\Pi}^1_n(X) = \bigcup_n \mathbf{\Delta}^1_n(X)).$$

We call $\{\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n\}$ the *projective hierarchy.* It is proper, i.e., $\mathbf{\Sigma}^1_n \neq \mathbf{\Pi}^1_n, \mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subsetneqq \mathbf{\Delta}^1_{n+1}$, if X is uncountable.

In descriptive set theory one studies the structure of sets in these hierarchies (and even more extended ones). We will only need to consider in these lectures Borel sets and sets in the 1st level of the projective hierarchy, i.e., analytic ($\mathbf{\Sigma}^1_1$) and co-analytic ($\mathbf{\Pi}^1_1$) sets.

These are tied up by the well-known *Souslin Theorem*

$$\mathbf{B} = \mathbf{\Delta}^1_1.$$

(It is not hard to check that $\mathbf{B} \subseteq \mathbf{\Delta}^1_1$; it is the inclusion that $\mathbf{\Delta}^1_1 \subseteq \mathbf{B}$ that is the main point here.) So if a set is both analytic and co-analytic, it is Borel. However, there are, in every uncountable Polish space X, analytic (and so co-analytic sets) which are not Borel. One of the early examples is due to Hurewicz: If X is an uncountable compact metric space, then $K_\omega(X) = \{K \in K(X)\colon K \text{ is countable}\}$ is $\mathbf{\Pi}^1_1$ but not Borel. Another early example is due to Mazurkiewicz: In $C(\mathbb{T})$, the set $\{f \in C(\mathbb{T})\colon f \text{ is differentiable}\}$ is $\mathbf{\Pi}^1_1$ but not Borel.

If a set A, in a given space X, is not Borel, then this implies that one cannot give a necessary and sufficient criterion for membership in A, i.e., a characterization of membership in A, which is simple enough to be expressible in terms of countable operations starting from the basic information describing the members of X. So such a fact about the descriptive complexity of A gives important information about possible characterizations of membership in A. We want to apply this descriptive approach to the (closed) sets of uniqueness.

For that purpose it will be useful to first discuss another example of a co-analytic non-Borel set. First we recall a standard fact from the theory of analytic sets.

17.1. Theorem. *For every Polish space X and $\mathbf{\Sigma}^1_1$ set $A \subseteq X$, there is a G_δ set $G \subseteq X \times \mathcal{C}$, where $\mathcal{C} = 2^{\mathbb{N}}$ is the Cantor space, such that $A = \mathrm{proj}_X[G]$, i.e.,*

$$x \in A \Leftrightarrow \exists y \in 2^{\mathbb{N}} (x, y) \in G.$$

Proof. This is clear if $A = \emptyset$, so we assume that $A \neq \emptyset$. The nonempty analytic sets can be also characterized as the continuous images of Polish spaces and since every Polish space is the continuous image of the *Baire space* $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ (which is homeomorphic to the irrationals), it follows that there is continuous $g : \mathcal{N} \longrightarrow X$ with $g[\mathcal{N}] = A$. Let $G \subseteq X \times \mathcal{N}$ be defined by

$$(x, y) \in G \Leftrightarrow f(y) = x.$$

Then $A = \mathrm{proj}_X[G]$. So A is the projection of a closed subset of $X \times \mathcal{N}$. But it is easy to see that $\mathcal{N}$ is (homeomorphic to) a G_δ subset of $\mathcal{C}$. (View $\mathcal{C}$ as $2^{\mathbb{N}\times\mathbb{N}}$ and identify $x \in \mathbb{N}^{\mathbb{N}}$ with its graph which is a subset of $\mathbb{N} \times \mathbb{N}$.) So G, viewed as a subset of $X \times \mathcal{C}$, is G_δ in $X \times \mathcal{C}$ and we are done. $\dashv$

We now have:

17.2. Theorem (Hurewicz). *Let $\mathbb{Q}' = \mathbb{Q} \cap [0, 1]$. Then $K(\mathbb{Q}')$ is $\mathbf{\Pi}^1_1$ but not Borel in $K([0, 1])$.*

Proof. Note first that $K(\mathbb{Q}')$ is $\mathbf{\Pi}^1_1$: Put $N = [0,1] \setminus \mathbb{Q}'$. Then

$$\begin{aligned} \sim K(\mathbb{Q}') &= \{K \in K([0,1]) : K \cap N \neq \emptyset\} \\ &= \{K \in K([0,1]) : \exists x (x \in K \ \& \ x \in N)\} \\ &= \ \mathrm{proj}_{K([0,1])}[G], \end{aligned}$$

where $G \subseteq K([0,1]) \times [0,1]$ is defined by

$$(K, x) \in G \Leftrightarrow x \in K \ \& \ x \in N.$$

So G is G_δ and thus $\sim K(\mathbb{Q}')$ is $\mathbf{\Sigma}^1_1$, and $K(\mathbb{Q}')$ is $\mathbf{\Pi}^1_1$.

To show that $K(\mathbb{Q}')$ is not Borel, we will first work with $K(Q)$, where $Q \subseteq \mathcal{C} = 2^{\mathbb{N}}$ is the countable dense consisting of the eventually periodic sequences. We will show that $K(Q)$ is not Borel (in $K(\mathcal{C})$). Granting this, we complete the proof as follows: Let $f : \mathcal{C} \to [0,1]$ be defined by $f(x) = \sum_{n=0}^{\infty} x(n) 2^{-n-1}$. Then f is continuous and $x \in Q \Leftrightarrow f(x) \in \mathbb{Q}'$. Let $F : K(\mathcal{C}) \to K([0,1])$ be defined by $F(K) = f''(K) = f[K]$. Then F is continuous and

$$K \in K(Q) \Leftrightarrow F(K) \in K(\mathbb{Q}'),$$

so $K(Q) = F^{-1}(K(\mathbb{Q}'))$. If $K(\mathbb{Q}')$ was Borel, then $K(Q)$ would be Borel too, being a continuous preimage of a Borel set, a contradiction.

So it is enough to show $K(Q)$ is not Borel in $K(\mathcal{C})$. We will, in fact, prove that this holds for any countable dense set $Q \subseteq \mathcal{C}$. This is based on the following lemma.

Lemma. *Let $F \subseteq 2^{\mathbb{N}}$ be F_σ. Then there is a continuous function $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $F = g^{-1}(Q)$.*

Assuming this, we complete the proof as follows: Fix a $\mathbf{\Pi}^1_1$ not Borel set $P \subseteq \mathcal{C}$. Then, by 17.1, let F be F_σ in $\mathcal{C} \times \mathcal{C}$ such that

$$x \notin P \Leftrightarrow \exists y (x, y) \notin F$$

or

$$x \in P \Leftrightarrow \forall y (x, y) \in F.$$

Now $\mathcal{C} \times \mathcal{C}$ is homeomorphic to $\mathcal{C}$, so that there is continuous $g : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with $g^{-1}[Q] = F$. Let $G : \mathcal{C} \to K(\mathcal{C})$ be defined by

$$G(x) = g(\{x\} \times \mathcal{C}).$$

Then G is continuous and

$$\begin{aligned} x \in P &\Leftrightarrow \forall y (x,y) \in F \\ &\Leftrightarrow \forall y (g(x,y) \in Q) \\ &\Leftrightarrow G(x) \subseteq Q \\ &\Leftrightarrow G(x) \in K(Q). \end{aligned}$$

So, as before, if $K(Q)$ was Borel, so would be P, a contradiction.

Proof of the Lemma. Consider the following game: In a run of the game, Players I,II take turns (I starting first) choosing successively $x(0), y(0), x(1), y(1), \cdots ; x(i), y(i) \in \{0,1\}$. II wins iff $x \in F \Leftrightarrow y \in Q$. A *strategy* for II is a map $\sigma : \{0,1\}^{<\mathbb{N}} \to \{0,1\}$. II follows this strategy in a run of this game if for each $n, y(n) = \sigma(x|n)$ (where $x|n = (x(0), \cdots, x(n-1))$). It is a *winning strategy* if II wins every run of the game in which he follows σ. (Strategies for I are similarly defined.) Put $\sigma^*(x) = y$ iff $\forall n(y(n) = \sigma(x|n))$, i.e., $\sigma^*(x)$ is what II plays following σ, when I plays x in a given run of the game. Thus if σ is a winning strategy for II, clearly $x \in F \Leftrightarrow \sigma^*(x) \in Q$. Since easily $\sigma^* : \mathcal{C} \to \mathcal{C}$ is a continuous function, it is enough to show that II has a winning strategy in this game. We will define such a strategy below.

Let $F = \bigcup_n F_n$, F_n closed. Let for each closed set $H \subseteq \mathcal{C}, T_H = \{x|n : x \in H\}$. Then T_H is a tree on $\{0,1\}$, i.e., a subset of $\{0,1\}^{<\mathbb{N}}$ closed under initial segments (i.e., $s = (s_0, \cdots, s_{n-1}) \in T_H$ and $m < n$ implies that $(s_0, \cdots, s_{m-1}) \in T_H$). Moreover

$$[T_H] = \{x : \forall n(x|n \in T_H)\} = H.$$

Let $T_{F_n} = T_n$ and $Q = \{q_n\}$. Here is then the strategy for II:

As I plays $x(0), x(1), \cdots$ II plays $y(0), y(1), \cdots$ as follows: As long as I stays within T_0, i.e., $x|n \in T_0$, II plays $y(0) = q_0(0), y(1) = q_0(1), \cdots$, i.e., follows $q_0 \equiv q'_0$. If x ever gets out of T_0 let $n_0 + 1$ be least with $x|(n_0+1) \notin T_0$. Then II plays $y(n_0) \neq q_0(n_0)$ and chooses $q'_1 \in Q$ with $y|(n_0+1)$ an initial segment of q'_1. This can be done as Q is dense. From then on, if x stays within T_1, II follows q'_1. If x ever gets out of T_1, let $n_1 > n_0$ be least with $x|(n_1+1) \notin T_1$. Then II plays $y(n_1) \neq q_1(n_1)$ and chooses $q'_2 \in Q$ with $y|(n_1+1)$ an initial segment of q'_2, and so on ad infinitum. $\dashv$

§18. The theorem of Kaufman and Solovay.

I will prove here that the set U of closed sets of uniqueness is not Borel in the space $K(\mathbb{T})$. This result is due to Kaufman and Solovay independently. The proof that I will give is a simplification of Solovay's argument and is based on two facts about U: (1) Bari's Theorem that the countable union of closed sets of uniqueness is also a set of uniqueness and (2) the general form of the Salem-Zygmund. I will give the proof of Bari's Theorem after giving the proof of the Kaufman-Solovay result.

18.1. Theorem (Kaufman,Solovay). *The set U of closed sets of uniqueness is not Borel (in $K(\mathbb{T})$).*

Proof. In the notation of 15.2, let, for each $x \in [0,1]$,

$$f(x) = E(1/4; \frac{3}{8} + \frac{x}{9}, 3/4).$$

Then $f(x)$ is a perfect set in $\mathbb{T}$, so $f : [0,1] \to K(\mathbb{T})$.

18.2. Exercise. f is continuous.

From 15.2 it now follows that

$$x \in \mathbb{Q} \Leftrightarrow f(x) \in U.$$

Let $F : K([0,1]) \to K(\mathbb{T})$ be defined by

$$F(K) = \bigcup f''(K)$$

$$= \bigcup \{f(x) : x \in K\}.$$

Then F is continuous and

$$K \subseteq \mathbb{Q} \Leftrightarrow F(K) \in U,$$

since the union of countably many closed sets of uniqueness is a set of uniqueness. Thus, in the notation of 17.2,

$$K(\mathbb{Q}') = F^{-1}(U),$$

so U cannot be Borel, since $K(\mathbb{Q}')$ is not Borel. $\dashv$

In 15.2 it is actually proved that if $1/\xi$ is not a Pisot or else one of $\eta_1, \cdots, \eta_k$ is not in $\mathbb{Q}(\xi)$, then not only $E(\xi; \eta_1, \cdots, \eta_k) \notin U$ but also $E(\xi; \eta_1, \cdots, \eta_k) \notin U_0$. So it follows, in the notation of the preceding proof, that

$$K(\mathbb{Q}') = F^{-1}(U_0),$$

so we also have:

18.3. Corollary (of the proof). *U_0 is not Borel.*

Remark. Another proof of (much stronger versions of) 18.2 and 18.3, which is self-contained and independent of the Salem-Zygmund theorem, will be given in §27 below.

Since the sets $F(K)$ are also perfect it finally follows that:

18.4. Corollary. *The class of perfect sets of uniqueness is not Borel (in $K(\mathbb{T})$).*

This result has obvious implications for the Characterization Problem: It is impossible to characterize, given a standard description of a perfect set (e.g., in terms of the sequence of its contiguous intervals), whether it is a set of uniqueness, by conditions which are explicit enough to be expressed by countable operations involving this description. This is because any such description would give a Borel definition of the set U in $K(\mathbb{T})$. It should be noted that all positive results obtained so far, including the Salem-Zygmund Theorem and its generalizations, are of this nature. Later on we will see some even stronger conclusions ruling out even more general types of characterization (see §24).

§19. Descriptive classification as a method of existence proof.

We describe here a method of existence proof based on the concept of descriptive classification of sets in Polish spaces. Suppose we have two properties R, S and every object satisfying R satisfies also S. Our problem is to find an object satisfying S but not R. The *descriptive method* consists of finding an appropriate Polish space X and calculating the descriptive complexity of $\{x \in X : R(x)\} = R^*, \{x \in X : S(x)\} = S^*$. Clearly $R^* \subseteq S^*$. If for instance R^* turns out to be of descriptive complexity different than S^*, e.g., if R^* is non-Borel but S^* is Borel, or R^* is not $\mathbf{\Sigma}^1_1$ but S^* is $\mathbf{\Sigma}^1_1$, then clearly $R^* \subsetneqq S^*$, thus $\exists x \in X(S(x) \ \& \ \neg R(x))$, so we have shown the existence of an object satisfying S but not R.

Here is an example (due to Bourgain) of the application of this method: Given a class $\mathcal{S}$ of separable Banach spaces, a separable Banach space X is *universal* for $\mathcal{S}$ if every $Y \in \mathcal{S}$ is isomorphic to a closed subspace of X, i.e., can be embedded into X. An old problem in the theory of Banach spaces asked whether there is a separable Banach space with separable dual which is universal for the class of separable Banach spaces with separable dual (Problem 49 in the Scottish book, Mauldin [1981]). This was answered negatively by Wojtaszczyk. Bourgain then showed that any separable Banach space universal for the above class must be universal for the class of all Banach spaces (so it cannot have separable dual). The method of proof is the following: Let X_0 be universal for the class of separable Banach spaces with separable dual. Then one can calculate that

$$S = \{K \in K(\mathcal{C}) : C(K) \text{ is isomorphic to a closed subspace of } X_0\}$$

is $\mathbf{\Sigma}^1_1$. Let

$$R = \{K \in K(e) : K \text{ is countable }\}.$$

Then, by a result of Hurewicz (see §22 below), R is $\mathbf{\Pi}^1_1$ but not Borel, so not $\mathbf{\Sigma}^1_1$ (by Souslin's Theorem). Now

$$R \subseteq S$$

(as the dual of $C(K)$ is the space $M(K)$, which, as K is countable, is easily separable). So $R \neq S$ and there is $K \in K(C), K$ uncountable with $C(K)$ isomorphic to a closed subspace of X_0. But, as K is uncountable, $C(K)$ is universal for all separable Banach spaces, and thus so is X_0.

19.1. Exercise. Show that $\{K \in K(\mathbb{T}) : \lambda(K) = 0\}$ is G_δ in $K(\mathbb{T})$. Use this and *just the statement* of 18.1 to deduce Menshov's Theorem (that there is a closed null set of multiplicity).

§20. Bari's Theorem.

I will now prove the following important result of Bari that was used in the proof of 18.1.

20.1. Bari's Theorem. *The union of countably many closed sets of uniqueness is a set of uniqueness.*

Proof. We will need the following result of de la Vallée-Poussin (1912), which should be contrasted with Menshov's Theorem that a trigonometric series can converge a.e. without being identically 0.

20.2. Theorem (de la Vallée-Poussin). *Let $S \sim \sum c_n e^{inx}$ be a trigonometric series such that for each x, there is $M_x < \infty$ with $|\sum_{n=-N}^{N} c_n e^{inx}| \leq M_x$ for all $N \geq 0$ (i.e., $\sum c_n e^{inx}$ has bounded partial sums). Then if $\sum c_n e^{inx} = 0$ a.e., we have that $c_n = 0, \forall n$.*

I will postpone for a while the proof of 20.2 and use it to prove Bari's Theorem. So assume $E_n \subseteq \mathbb{T}$ are closed and $E_n \in U$. Put $E = \bigcup_n E_n$. Let $\sum c_n e^{inx} = 0$ off E, in order to show that $c_n = 0, \forall n$. Clearly $\lambda(E_n) = 0$, so $\lambda(E) = 0$, thus $c_n \to 0$. Assume c_n is not identically 0, towards a contradiction. Let

$$G = \{x : \{S_N(x)\} \text{ is unbounded}\},$$

where

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}.$$

Then $G \subseteq E$, G is G_δ and $G \neq \emptyset$ by de la Vallé-Poussin's Theorem. So G is Polish in the relative topology and if $E_n \cap G = G_n$, clearly G_n is closed in G and $\bigcup_n G_n = G$, so, by the Baire Category Theorem, there is an open interval I_0 and some n_0 with $G \cap I_0 = G_{n_0} \cap I_0 \neq \emptyset$. We will show that $\sum c_n e^{inx} = 0$ on I_0, thus $I_0 \cap G = \emptyset$, so we have a contradiction.

We use Rajchman multiplication. Choose $f \in C(\mathbb{T})$ infinitely differentiable with $f > 0$ on I_0 and $f = 0$ off I_0. Let $T = S(f) \cdot S, T \sim \sum C_n e^{inx}$. Recall that, by 13.2, $\sum_{-\infty}^{\infty} (C_n - f(x)c_n)e^{inx} = 0, \forall x$. If we can show that $C_n = 0, \forall n$, then we are done. As E_{n_0} is a set of uniqueness, it is enough to show that $\sum C_n e^{inx} = 0$ for $x \notin E_{n_0}$. So let $x \notin E_{n_0}$. We can assume that $x \in I_0 \cap E$, since $\sum C_n e^{inx} = 0$ off $I_0 \cap E$. So let $J_0 \subseteq I_0$ be an interval containing x such that $\overline{J} \cap E_{n_0} = \emptyset$.

Choose again an infinitely differentiable $g \in C(\mathbb{T})$ with $g(x) = 1$ and $\text{supp}(g) \subseteq \overline{J}$. Again if

$$R \sim S(g) \cdot T,\ R \sim \sum D_n e^{inx},$$

$\sum D_n e^{inx} = 0$ a.e., because $\sum C_n e^{inx} = 0$ a.e. (as $\sum c_n e^{inx} = 0$ a.e.), and has bounded partial sums outside $\overline{J_0} \cap G = \overline{J_0} \cap G_{n_0} = \emptyset$ (since $\sum c_n e^{inx}$, and thus $\sum C_n e^{inx}$, has the same property outside G), i.e., $\sum D_n e^{inx}$ has bounded partial sums everywhere, so by de la Vallé-Poussin again, $D_n = 0, \forall n$, and, by 13.2, $\sum C_n e^{inx} = 0$.

Proof of 20.2.

Lemma. *Let $G \subseteq [0, 2\pi]$ be G_δ and null. Then there is $g \geq 0$ continuous nondecreasing in $[0, 2\pi]$ with $g'(x) = +\infty$ for $x \in G$.*

Proof. Let $G = \bigcap_n G_n$, G_n open, $\lambda(G_n) < 2^{-n}$. Let $g_n(x) = \frac{1}{2\pi}\int_0^x \chi_{G_n}(t)dt$. Then $0 \leq g_n \leq 2^{-n}$. Let $g = \sum g_n$. It is enough to show that $g'(x) = \infty$ for $x \in G$. Fix $K > 0$. For any $n_0 > K$, let $\epsilon > 0$ be such that $(x - \epsilon, x + \epsilon) \subseteq G_0 \cap \cdots \cap G_{n_0}$. Then if $0 < |h| < \epsilon$,

$$\frac{g(x+h) - g(x)}{h} \geq \frac{(n_0 + 1)h}{2\pi \cdot h} > \frac{k}{2\pi},$$

so $g'(x) = +\infty$. $\dashv$

Now assume $\sum c_n e^{inx} = 0$ a.e., and $\sum c_n e^{inx}$ has bounded partial sums at each point x. Let G be a null G_δ set with

$$x \notin G \Rightarrow \sum c_n e^{inx} = 0.$$

Let g be as in the preceding lemma. Put

$$f(x) = \int_0^x g(t)dt + C,$$

where $C < 0$ is chosen so that $f(2\pi) = 0$. So $f(x)$ is convex and $f'(x) = g(x)$.

Let F_S be the Riemann function of S and choose a, b so that if

$$F(x) = F_S(x) + ax + b,$$

then $F(0) = F(2\pi) = 0$. If we can show that $F = 0$ on $[0, 2\pi]$, then F_S is linear on $[0, 2\pi]$, so $c_n = 0, \forall n$.

We will show that $F \geq 0, F \leq 0$ on $[0, 2\pi]$:

$F \leq 0$: For $\epsilon > 0, x \in [0, 2\pi]$ let

$$F_\epsilon(x) = F(x) - \epsilon x(2\pi - x) + \epsilon f(x).$$

Then $F_\epsilon(0) < 0$, $F_\epsilon(2\pi) = 0$, so if $F \leq 0$ fails, towards a contradiction, there is $\epsilon > 0$ and $x_0 \in (0, 2\pi)$ at which F_ϵ achieves a maximum which is positive. Then for small enough h

$$0 \geq \frac{\Delta^2 F_\epsilon(x_0, h)}{h^2} = \frac{\Delta^2 F(x_0, h)}{h} + 2\epsilon + \frac{\epsilon \Delta^2 f(x_0, h)}{h^2}.$$

Now consider 2 cases:

Case 1. $x_0 \notin G$. Then $\sum c_n e^{inx_0} = 0$, so $D^2 F_S(x_0) = D^2 F(x_0) = 0$, thus $\frac{\Delta^2 F_S(x_0,h)}{h^2} \to 0$ as $h \to 0$. But, f being convex, $\Delta^2 f(x_0, h) \geq 0$, so we have a contradiction.

Case 2. $x_0 \in G$. Since $\{S_N(x_0)\}$ is bounded, we claim that

$$\left| \frac{\Delta^2 F(x_0, h)}{h^2} \right| < K < \infty \tag{$*$}$$

But $D^2 f(x_0) = g'(x_0) = +\infty$, so again we have a contradiction.

We can see $(*)$ as follows: we have

$$\frac{\Delta^2 F(x_0, 2h)}{4h^2} = \frac{\Delta^2 F_S(x_0, 2h)}{4h^2}$$

$$= \sum_{n \in \mathbb{Z}} \left(\frac{\sin nh}{nh} \right)^2 c_n e^{inx_0}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\sin nh}{nh} \right)^2 (a_n \cos nx_0 + b_n \sin nx_0).$$

We know that the partial sums

$$S_N(x_0) = \sum_{n=0}^{N} (a_n \cos nx_0 + b_n \sin nx_0)$$

$$= \sum_{n=-N}^{N} c_n e^{inx_0}$$

are bounded, say in absolute value by M. But we have

$$\sum_{n=0}^{\infty} \left(\frac{\sin nh}{nh} \right)^2 (a_n \cos nx_0 + b_n \sin nx_0)$$

$$= \sum_{n=0}^{\infty} S_N(x_0) \left(\left(\frac{\sin nh}{nh} \right)^2 - \left(\frac{\sin(n+1)h}{(n+1)h} \right)^2 \right),$$

so

$$\left| \frac{\Delta^2 F(x_0, 2h)}{4h^2} \right| \leq M \cdot \sum_{n=0}^{\infty} \left| \left(\frac{\sin nh}{nh} \right)^2 - \left(\frac{\sin(n+1)h}{(n+1)h} \right)^2 \right|$$

$$\leq M \cdot C = K < \infty,$$

where

$$C = \int_0^\infty |u'(x)|dx,$$

with $u(x) = (\sin x/x)^2$ (see 2.5).

$F \geq 0$: Replace ϵ by $-\epsilon$ in the above argument. $\dashv$

It follows from Bari's Theorem that the union of countably many F_σ sets of uniqueness is a set of uniqueness. It is not known if the union of even two G_δ sets of uniqueness is a set of uniqueness.

§21. Computing the exact descriptive complexity of U, U_0.

We have seen that U, U_0 are not Borel sets. We will now compute that they are both $\mathbf{\Pi}^1_1$, and since then they are not $\mathbf{\Sigma}^1_1$, this determines the exact descriptive complexity of U, U_0. If one looks at the definition of U, U_0, a rather straightforward calculation shows that U, U_0 must be $\mathbf{\Pi}^1_2$. However, this is a very crude estimate and with some more work, which is based on some appropriate generalizations of the nontrivial Theorem 7.6, we can bring the complexity down to $\mathbf{\Pi}^1_1$, which is the exact level.

Given a closed set $K \in K(\mathbb{T})$, and a trigonometric series $S \sim \sum c_n e^{inx}$, with $|c_n|$ bounded, we reformulate the condition "$\sum c_n e^{inx} = 0$ off K" in functional analytic terms. The key is to identify S with the element of $\ell^\infty = (\ell^1)^*$ given by $\{c_n\}$. So from now on we will view trigonometric series $S \sim \sum c_n e^{inx}$ with bounded coefficients as elements of ℓ^∞ and simply write $c_n = S(n)$.

Next we view an element λ_n of ℓ^1 as identified with the function $f(x) = \sum \lambda_n e^{inx}$. Note that $\hat{f}(n) = \lambda_n$. These are of course exactly the functions with absolutely convergent Fourier series, and their class is traditionally denoted by $A(\mathbb{T}) = A$. Thus we let $A \equiv \ell^1$. Under this identification the element e_n of ℓ^1 (where $e_n(j) = 1$, if $j = n, e_n(j) = 0$, if $j \neq n$) is identified with e^{inx}.

Now each $S \in \ell^\infty$ operates on $f = \sum c_n e^{inx} \in A$ by

$$\langle f, S\rangle = \sum \hat{f}(n)S(-n).$$

Thus $S(n) = \langle e^{inx}, S\rangle$. In particular, if $\mu \in M(\mathbb{T})$ and $S = \hat{\mu}$, then $S \in \ell^\infty$ (as $|\hat{\mu}(n)| \leq ||\mu||_M < \infty$) and

$$\langle f, \hat{\mu}\rangle = \sum \hat{f}(n)S(-n) = \int f d\mu$$

for $f \in A$. So we can view $\langle f, S\rangle$ as some kind of generalized integral and S as some kind of generalized measure (operating though only on functions in A) and thus it is customary to call elements of ℓ^∞ *pseudomeasures* and write PM instead of ℓ^∞,

$$\mathrm{PM} = \ell^\infty.$$

Finally, every $f \in L^1(\mathbb{T})$ gives rise to $\hat{f} \in c_0$, so that it is customary to think of the elements of c_0 as generalized functions and thus call them *pseudofunctions* and write PF instead of c_0,

$$\mathrm{PF} = c_0.$$

So we have $\mathrm{PF}^* = A, A^* = \mathrm{PM}$ (and also PF is a closed subspace of PM).

Using this idea, we can define what it means to say that a closed set $K \subseteq \mathbb{T}$ *supports* a *pseudomeasure* S.

Definition. Let $S \in \mathrm{PM}$ and $K \in K(\mathbb{T})$. Then K *supports* S iff for any open interval $I \cap E = \emptyset$ and infinitely differentiable $\varphi \in C(\mathbb{T})$ supported by I (i.e., $\mathrm{sup}_P(\varphi) \subseteq I$), we have $\langle \varphi, S \rangle = 0$. (Note that $\varphi \in A$ too.)

21.1. Exercise. Show that if $\mu \in P(\mathbb{T}), E \in K(\mathbb{T})$, then E supports μ iff $\mu(E) = 1$.

Remark. One can easily see that if S is supported by K, then actually for any interval I disjoint from K and any $f \in A$ supported by I we also have $\langle f, S \rangle = 0$. To see this fix $\epsilon > 0$ and let φ be infinitely differentiable with $\varphi = 1$ on $\mathrm{sup}_P(f)$ and $\mathrm{sup}_P(\varphi) \subseteq I$. Letting for $f \in A$

$$||f||_A = ||\hat{f}||_{\ell^1} = \sum |\hat{f}(n)|,$$

let P be a trigonometric polynomial with $||f - P||_A < \epsilon/||\varphi||_A$. Then noting that A is actually a Banach algebra under pointwise multiplication, i.e., for $f, g \in A$, $fg \in A$ and $||fg||_A \leq ||f||_A ||g||_A$, we have that

$$||f - P\varphi||_A = ||f\varphi - P\varphi||_A$$

$$\leq ||f - P||_A \cdot ||\varphi||_A < \epsilon.$$

Thus f can be approximated in the norm of A by infinitely differentiable functions ψ with support contained in I, so $\langle f, S \rangle = 0$, as $\langle \psi, S \rangle = 0$ for any such ψ.

We now have the following generalization of 7.6.

21.2. Theorem. *Let $K \in K(\mathbb{T})$ be a closed set and let $S \in PF$. Then the following are equivalent:*

(i) S is supported by K,

(ii) $\sum S(n)e^{inx} = 0$ off K.

We will postpone the proof of this for a while. From 21.2 it immediately follows that, letting

$$M = K(\mathbb{T}) \setminus U, M_0 = K(\mathbb{T}) - U_0$$

be the classes of closed sets of multiplicity and *restricted multiplicity,* resp., we have

21.3 Corollary. *Let $E \in K(\mathbb{T})$. Then*

(i) $E \in M \Leftrightarrow \exists S \in PM(||S||_\infty \leq 1$ & $S \in PF$ & $S \neq 0$ & E supports $S)$,

(ii) $E \in M_0 \Leftrightarrow \exists \mu \in M(\mathbb{T})(||\mu||_M \leq 1 \;\&\; \mu \in PF \;\&\; E$ *supports* $\mu)$.

Using this one can easily prove the following.

21.4. Theorem. *The sets* U, U_0 *are* $\mathbf{\Pi}^1_1$.

Proof. It is clearly enough to show that M, M_0 are $\mathbf{\Sigma}^1_1$.

First consider M. By 21.3 (i) the set $M \subseteq K(\mathbb{T})$ is the projection of the following set in $K(\mathbb{T}) \times B_1$ (PM), where B_1 (PM) is the unit ball of PM ($= \ell^\infty$) with the weak*-topology:

$$P = \{(K, S) \in K(\mathbb{T}) \times B_1(\text{PM}) : \lim_{|n| \to \infty} |S(n)| = 0 \;\&\; S \neq 0 \;\&\; K \text{ supports } S\}.$$

Since for each n, the map $S \mapsto S(n) = \langle e^{-inx}, S\rangle$ is continuous in B_1 (PM), the condition $\lim_{|n| \to \infty} |S(n)| = 0 \;\&\; S \neq 0$ is clearly Borel. We next claim that

$$Q = \{(K, S) \in K(\mathbb{T}) \times B_1(\text{PM}) : K \text{ supports } S\}$$

is closed, which shows that P is Borel, so M is $\mathbf{\Sigma}^1_1$.

To see that S is closed, let $K_i, S_i \in Q$ and $K_i \to K$ (in the Hausdorff metric or, equivalently, the Vietoris topology) and $S_i \to S$ (in the weak*-topology, i.e., $\langle f, S_i\rangle \to \langle f, S\rangle$ for each $f \in A$). Now take an interval I disjoint from K and infinitely differentiable φ supported by I, in order to show that $\langle \varphi, S\rangle = 0$.

Let $J \subseteq I$ be a closed interval containing $\text{supp}(\varphi)$. Then if $U = \mathbb{T} \setminus J$ we have $K \subseteq U$, so, by the definition of the Vietoris topology, $K_n \subseteq U$ for all large enough n. Thus φ is supported by an interval disjoint from K_n, and so $\langle \varphi, S\rangle = 0$ and we are done.

The proof for M_0 is similar, with $B_1(M(\mathbb{T}))$ instead of B_1 (PM). ⊣

We finally give the proof of 21.2:

Proof of 21.2. (i) $\Rightarrow$ (ii) By the Riemann Localization Principle it is enough to show that F_S is linear in each open interval I disjoint from K. Let $a \in I$ and choose h small enough so that $\psi_{a,h}$, as defined in §12, is supported by I. Then

$$0 = \langle \psi_{a,h}, S\rangle = \sum \hat{\psi}_{a,h}(-n) S(n)$$

$$= \frac{\Delta^2 F_S(a, h)}{h^2},$$

so F_S is linear on I.

(ii) $\Rightarrow$ (i). Let I be an open interval disjoint from K and let $\varphi \in C(\mathbb{T})$ be infinitely differentiable supported by I. We want to show that

$$\langle \varphi, S\rangle = \sum \hat{\varphi}(n) S(-n) = 0.$$

Consider the formal product

$$S(\varphi) \cdot S = T,$$

where $T(m) = \sum \hat{\varphi}(n)S(m-n)$. Then $T(0) = \sum \hat{\varphi}(n)S(-n) = \langle \varphi, S \rangle$, so it is enough to show that $T = 0$. But by the Rajchman multiplication theory

$$\sum (T(n) - \varphi(x)S(n))e^{inx} = 0,$$

so, as $\sum S(n)e^{inx} = 0$ on I and $\varphi(x) = 0$ off I, $\sum T(n)e^{inx} = 0$ for all x, so $T = 0$. ⊣

§22. Co-analytic σ-ideals of compact sets.

We summarize two key properties of the classes U, U_0.

(1) U, U_0 are $\mathbf{\Pi}^1_1$ (and not Borel).

(2) U, U_0 are σ-ideals in the following sense:

Definition. Let X be a compact metric space and $I \subseteq K(X)$ a class of closed sets in X. We say that I is a *σ-ideal of closed sets* if it satisfies the following two properties:

(i) $K \in I, L \subseteq K, L \in K(X) \Rightarrow L \in I$;

(ii) $K_n \in I, K \in K(X), K = \bigcup_n K_n \Rightarrow K \in I$.

Kechris, Louveau and Woodin undertook the general study of $\mathbf{\Pi}^1_1$ σ-ideals of closed sets in compact metric spaces. This theory turned out to have interesting applications to several problems concerning uniqueness sets. I will next discuss some of the main results of this theory and some of its applications.

Before getting into these though, I want to give some examples of σ-ideals.

Examples. (i) For $A \subseteq X$, let

$$K(A) = \{K \in K(X) : K \subseteq A\},$$

$$K_\omega(A) = \{K \in K(X) : K \subseteq A, K \text{ countable}\}.$$

If A is $\mathbf{\Pi}^1_1$, then $K(A)$ and $K_\omega(A)$ are $\mathbf{\Pi}^1_1$. The first is easy and the second follows from:

22.1. Exercise. The set $K_\omega(X)$ is $\mathbf{\Pi}^1_1$. *Hint.* Use the Cantor-Bendixson Theorem.

(ii) $I_{\text{meager}} = \{K \in K(X)$: K is meager (e.g. nowhere dense)$\}$ is a σ-ideal, which is in fact G_δ.

(iii) If μ is a Borel probability Borel measure on X, then $I_\mu = \{K \in K(X) : \mu(K) = 0\}$ is a σ-ideal, which is also G_δ.

(iv) More generally, if M is a class of Borel probability measures on X and $M \subseteq P(X)$ is $\mathbf{\Sigma}^1_1$ in $P(X)$ with the weak*-topology ($P(X)$ is defined as in §10, where we dealt with the special case $X = \mathbb{T}$), then

$$I_M = \{K \in K(\mathbb{T}) : \forall \mu \in M(\mu(K) = 0)\}$$

is a $\mathbf{\Pi}^1_1$ σ-ideal.

(v) U, U_0 are $\mathbf{\Pi}^1_1$ σ-ideals. Note that $U_0 = I_R$, where $R = \{\mu \in P(\mathbb{T})\colon\ \mu$ is a Rajchman measure$\}$.

The first result I will discuss is a surprising dichotomy which limits sharply the possible descriptive complexities of $\mathbf{\Pi}^1_1$ σ-ideals of closed sets.

22.2. The Dichotomy Theorem (Kechris-Louveau-Woodin). *Let I be a $\mathbf{\Pi}^1_1$ σ-ideal of closed sets in a compact metric space X. Then either I is G_δ or else it is not Borel (and thus $\mathbf{\Pi}^1_1$ but not $\mathbf{\Sigma}^1_1$).*

We can prove this theorem by applying the following result of Hurewicz. Below, if Y is a topological space and $C \subseteq Y$, we say that C is a *Cantor set* if C is homeomorphic to $\mathcal{C} = 2^{\mathbb{N}}$.

22.3. Theorem (Hurewicz). *Let Y be a Polish space and $A \subseteq Y$ a $\mathbf{\Pi}^1_1$ set. Then either A is G_δ or else there is a Cantor set $C \subseteq Y$ such that $C \cap A$ is countable dense in C.*

Note that exactly one of these possibilities must occur, since if A is G_δ, $C \cap A$ is G_δ in C and so cannot be countable dense in C.

Proof of 22.2. Let $Y = K(X)$ and assume $I \subseteq Y$ is not G_δ. Then by 22.3 there is a Cantor set $C \subseteq Y$ with $C \cap I$ countable dense in C. This means that there is a $\varphi : \mathcal{C} \to C$ and a countable dense set $Q \subseteq \mathcal{C}$, with $\varphi^{-1}[C \cap I) = Q$. Let $f : K(\mathcal{C}) \to K(X)$ be defined by $f(K) = \bigcup \varphi''(K) = \bigcup \varphi[K]$. Clearly f is continuous. Moreover we claim that

$$K \in K(Q) \Leftrightarrow f(K) \in I,$$

because if $K \subseteq Q$, then $\varphi[K] \subseteq I$ and since $\varphi[K]$ is countable, $\bigcup \varphi[K] = f(K) \in I$. Conversely if $K \not\subseteq Q$, and $x \in K \setminus Q$, then $\varphi(x) \in \varphi[K] \setminus \varphi[Q]$, so $\varphi(x) \in \varphi[K] \setminus I$, and, since $\bigcup \varphi[K] \supseteq \varphi(x)$, $f(K) = \bigcup \varphi[K] \notin I$.

We have seen in the proof of 17.2 that $K(Q) = f^{-1}[I]$ is not Borel, so I can't be Borel either. $\dashv$

This result distinguishes all $\mathbf{\Pi}^1_1$ σ-ideals of closed sets into two main categories according to descriptive complexity:

(1) The *simple* ones, which are G_δ. Examples include:

(a) $K(A)$, for $A \in G_\delta$;

(b) I_{meager};

(c) I_μ.

(2) The *complicated* ones, which are $\mathbf{\Pi}^1_1$ but not Borel. Examples include:

(a) $K(A)$, if A is not G_δ.

(*Proof.* Since $x \in A \Leftrightarrow \{x\} \in K(A)$, if A is not G_δ, $K(A)$ cannot be G_δ, so it must be complicated.)

(b) (Hurewicz) $K_\omega(A)$, if A contains a Cantor set (e.g., if $A = X$ and X is uncountable).

(*Proof.* Let $C \subseteq A$ be a Cantor set. Since $K_\omega(C) = K(C) \cap K_\omega(A)$, it is enough to show $K_\omega(C)$ is not Borel. If it was, then it would be G_δ. But notice that $K_\omega(C)$ is dense in $K(\mathcal{C})$, since it contains all the finite sets. But $K_p(\mathcal{C}) = \{K \in K(\mathcal{C})\colon K \text{ is perfect}\}$ is also dense in G_δ in $K(\mathcal{C})$, so it must intersect $K_\omega(C)$ by the Baire category theorem, which is a contradiction.)

(c) U, U_0.

22.4. Exercise. Show that $K_\omega(A)$ is complicated for *any* uncountable $A \subseteq X,\ A \in \mathbf{\Pi}^1_1$. (If A is countable, clearly $K_\omega(A) = K(A)$.) [*Hint.* Consider cases as A is Borel or not. Use the fact that every uncountable Borel set contains a Cantor set.]

Before I proceed I will say a few things about the proof of 22.3. A nice proof of 22.3 can be given using games. I will consider only the case when Y is actually compact metrizable.

First we can reduce the problem to $\mathcal{C}$. For this we use the fact that, since Y is compact metrizable, there is a continuous surjection $\varphi : \mathcal{C} \to Y$. Consider the $\varphi^{-1}[A] = A'$. Then A' is $\mathbf{\Pi}^1_1$. Assume the result has been proved for the space $\mathcal{C}$. Then either A' is G_δ or else there is a Cantor set $C' \subseteq \mathcal{C}$ with $C' \cap A'$ countable dense in C'.

In the first case $B' = \mathcal{C} \setminus A'$ is F_σ, so $\varphi[B'] = X \setminus A$ is F_σ (since we are working in compact spaces), so A is G_δ.

In the second case, let $C'' = \varphi[C']$. Then C'' is closed in Y, and $A \cap C''$ is countable dense in C''. Note also that $(Y \setminus A) \cap C''$ is dense in C'', so C'' is perfect. It is now easy, by a Cantor-type construction, to find a Cantor set $C \subseteq C''$ such that $C \cap A$ is countable dense in C. So it is enough to assume that $Y = \mathcal{C}$.

Fix a countable dense subset $Q \subseteq \mathcal{C}$ and consider the following game as in the proof of the lemma in 17.2: I plays $x \in \mathcal{C}$, II plays $y \in \mathcal{C}$ and II wins iff $x \in Q \Leftrightarrow y \in A$.

If I has a winning strategy, this gives a continuous function $f : \mathcal{C} \to \mathcal{C}$ such that $y \in A \Leftrightarrow f(y) \notin Q$, so $A = f^{-1}[\mathcal{C} \setminus Q]$, and since $\mathcal{C} \setminus Q$ is G_δ, so is A. If on the other hand II has a winning strategy, then this gives a continuous function g such that

$$x \in Q \Leftrightarrow g(x) \in A.$$

Let $g[\mathcal{C}] = K$. Then notice that K is closed and $g[Q], g[\mathcal{C} \setminus Q]$ are disjoint and dense in K, so K is perfect. Moreover $A \cap K = g[Q]$, so $A \cap K$ is countable dense in K. So again, by a simple Cantor-type construction, we can find a Cantor set $C \subseteq K$ with $A \cap C$ countable dense in C.

So if this game is determined, i.e., one of the players has a winning strategy, the proof is complete. The *payoff* set of this game, i.e., the set

$$\{(x, y) : x \in Q \Leftrightarrow y \in A\}$$

is a Boolean combination of $\mathbf{\Pi}_1^1$ sets (in $\mathcal{C}\times\mathcal{C}$), so by a theorem of Martin, the determinacy of this game follows from appropriate large cardinal axioms in set theory. However, we cannot prove the determinacy of such complex games in classical set theory (ZFC), since the best result provable in it is the determinacy of all Borel games (Martin). This problem can be handled by considering an appropriate modification of this game, which still does the job, and turns out to be Borel, in fact a Boolean combination of F_σ sets, so its determinacy can be established in classical set theory (see Kechris [1995], §21].

§23. Bases for σ-ideals.

Definition. Let I be a σ-ideal of closed sets in a compact metric space X. A *basis* for I is a subset $B \subseteq I$ such that B is *hereditary,* i.e., $K \in B, L \subseteq K, L \in K(X) \Rightarrow L \in B$, and B generates I, i.e., for any $K \in I, \exists\{K_n\} \subseteq B$ with $K = \bigcup_n K_n$.

We will consider here the question of whether a given σ-ideal admits a *Borel basis.* The motivation comes again from the Characterization Problem for U.

Although one cannot hope to find a very explicit characterization of when a closed set $K \subseteq \mathbb{T}$ is in U or not, it may still be possible to find a simple subclass B of U, like e.g., the H-sets that we considered in §14, so that every U-set can be written on a countable union of sets in B. Such questions have been raised in this subject periodically. For example, it was indeed considered whether every U set can be written as a countable union of H-sets or more generally a countable union of so-called $H^{(n)}$-sets, a generalization of H-sets ($H = H^{(1)}$)). The answer turned out to be negative in this case (Piatetski-Shapiro). The general philosophy is the following: Is it possible to understand U-sets as countable unions of some explicitly characterizable subclass? This can then be formalized as follows:

The Basis Problem. Does the σ-ideal U of closed uniqueness sets admit a Borel basis?

A negative answer would provide a much stronger limitative result concerning the characterization problem. But it would also be a powerful existence theorem (again a use of the descriptive method): Given any simply definable (i.e., Borel) hereditary collection of closed uniqueness sets B, there exists a $K \in U$ which cannot be written as a countable union of sets in B. For example, since the H-sets (and the $H^{(n)}$-sets) can be easily shown to form a Borel class, this would immediately imply the result of Piatetski-Shapiro quoted earlier. But instead of relying on ad-hoc constructions to deal with existence of such examples for any potentially proposed class B, a negative answer to the Basis Problem would once and for all deal with all such (reasonable) possibilities without such constructions.

A similar basis problem can of course be raised for the σ-ideal U_0.

Our main goal here is to develop a method for demonstrating non-existence of Borel bases. In fact, the main result that I will prove below establishes (under certain conditions) an important and quite strong property that all σ-ideals with Borel bases must necessarily have. This can be used to prove non-existence of Borel bases by showing that a given σ-ideal fails to have this strong property. This is how one shows that U has no Borel basis. But on the other hand, if it happens to be the case that one deals with an ideal that has a

Borel basis (and, as it turns out, U_0 is such an example) this establishes this very strong property. And this is how the original solution of the Category Problem came about.

Before I proceed I would like to mention a couple of examples.

Examples. (i) Every G_δ σ-ideal has a Borel basis (namely itself).

(ii) $K_\omega(X)$ has a Borel basis, namely $\{\emptyset\} \cup \{\{x\} : x \in X\}$.

(iii) $K(A)$ has no Borel basis, if A is $\mathbf{\Pi}^1_1$ but not Borel (since for any basis $B \subseteq K(A), x \in A \Leftrightarrow \{x\} \in B$). In fact it turns out $K(A)$ has a Borel basis iff A is the difference of two G_δ sets (Kechris-Louveau-Woodin).

For any hereditary $B \subseteq K(X)$ let

$$B_\sigma = \{K \in K(X) : K = \bigcup_n K_n, K_n \in B\},$$

be the σ-ideal generated by I. Thus B is a basis for I iff $I = B_\sigma$.

23.1. Exercise. For $B \subseteq K(X)$ hereditary and any $K \in K(X)$ define the *B-derivative* K'_B of F by

$$K'_B = \{x \in K : \forall \text{ open } V(x \in V \Rightarrow \overline{K \cap V} \notin B)\}.$$

(Notice that for $B = \{\emptyset\} \cup \{\{x\} : x \in B\}, K'_B = K'$). Then by transfinite induction define $K^{(0)}_B = K, K^{(\alpha+1)}_B = (K^{(\alpha)}_B)', K^{(\lambda)}_B = \bigcap_{\alpha<\lambda} K^{(\alpha)}_B$. There is again a countable ordinal α_0 such that $K^{(\alpha_0)}_B = K^{(\beta)}_B, \forall \beta \geq \alpha_0$. The least one is denoted by $rk_B(K)$ and called the *Cantor-Bendixson rank of K associated to B.* Put $K^{(\infty)}_B = K^{(rk_B(K))}_B$. Show that

$$K \in B_\sigma \Leftrightarrow K^{(\infty)}_B = \emptyset.$$

Call $K \in K(X)$ *B-perfect* if $K'_B = K$, i.e., $\forall$ open $V(K \cap V \neq \emptyset \Rightarrow \overline{K \cap V} \notin B)$. Show the analog of Cantor-Bendixson, namely that any K can be uniquely written as

$$K = P \cup C$$

with P *B-perfect* and C contained in a countable union of sets in B. Show that $P = K \setminus \bigcup\{V \text{ open: } \overline{K \cap V} \in B_\sigma\} = K^{(\infty)}_B$.

Show that if B is $\mathbf{\Pi}^1_1$, then

$$\{K \in K(X) : K \text{ is } B\text{−perfect}\}$$

in $\mathbf{\Sigma}^1_1$. [*Hint.* Show that $f : Y \to K(X)$ is Borel iff for any open $W \subseteq Y, \{y : f(y) \cap W \neq \emptyset\}$ is Borel. Use this to show that $K \mapsto \overline{K \cap V}$ is Borel, for any open $V \subseteq X$.]

Conclude that if $B \subseteq K(X)$ is hereditary $\mathbf{\Pi}^1_1$, then B_σ is also $\mathbf{\Pi}^1_1$.

(In particular, this shows that only $\mathbf{\Pi}^1_1$ σ-ideals can have Borel (in fact even $\mathbf{\Pi}^1_1$) bases.)

If B is Borel show that the map

$$K \mapsto K'_B$$

is Borel. Thus one has a "semi-Borel" test for membership in B_σ : $K \in B_\sigma$ iff the transfinite iteration of the Borel operation $K \mapsto K'_B$ terminates after countably many steps (depending on K) with the empty set.

Remark. It can be also shown that the following are equivalent for any $\mathbf{\Pi}^1_1$ σ-ideal I:

(i) I admits a Borel basis,

(ii) I admits a $\mathbf{\Sigma}^1_1$ basis.

(iii) There is $B \subseteq I$ (not necessarily hereditary), $B \in \mathbf{\Sigma}^1_1$, such that $I = \{K \in K(X)$: $\exists \{K_n\} \subseteq B(K \subseteq \bigcup_n K_n)\}$.

Before I state the main result I need one more definition.

Definition. A σ-ideal I of closed sets is *calibrated* if for any closed set F, and any sequence $F_n \in I$ if $K(F \setminus \bigcup_n F_n) \subseteq I$, then $F \in I$.

Examples. (i) I_{meager} (in any perfect X) is *not* calibrated. Because if $\{x_n\}$ is dense, $F_n = \{x_n\} \in I_{\text{meager}}, K(X \setminus \bigcup_n F_n) \subseteq I_{\text{meager}}$, but $X \notin I_{\text{meager}}$.

(ii) $K_\omega(X)$ is calibrated (since every uncountable G_δ set contains an uncountable closed set).

(iii) If $M \subseteq P(X)$, then I_M is calibrated (since for every $\mu \in P(X)$, every Borel set of positive μ-measure contains a closed set of positive μ-measure).

Thus calibration can be thought of as a (weak) generalization of the idea of inner regularity of measures.

We now have the following result.

23.2. The Basis Theorem (Kechris-Louveau-Woodin) *Let I be a calibrated σ-ideal of closed subsets of X. Assume I admits a non-trivial basis, in the local sense that there is a basis $B \subseteq I$ such that for every open $\emptyset \neq V \subseteq X$, there is $K \subseteq V$, $K \in I \setminus B$. Then if $A \subseteq X$ has the Baire property and $K(A) \subseteq I$ (i.e., every closed subset of A is in I), then A is meager.*

Proof. Assume A is not meager. Then, as A has the property of Baire, there is an open set $U \neq \emptyset$ on which A is comeager, so A contains a G_δ set G which is dense in U. We will derive a contradiction by showing that there is $K \notin I$, $K \in K(X), K \subseteq G$.

To simplify the notation, we will also assume that $U = X$ (otherwise we can do the construction below within U). So the context is the following:

$G \subseteq X$ is dense G_δ and we want to construct $K \in K(X)$, $K \notin I, K \subseteq G$.

Notice first that every $K \in I$ is meager: Otherwise $K = \bigcup_n K_n, K_n \in B$, and so for some $K_n \in B$, K_n is non-meager, so there is non-$\emptyset$ open V with $V \subseteq K_n$. Thus

$$I \cap K(V) = B \cap K(V)(= K(V)),$$

a contradiction.

Notice also that if $K \in K(X)$ is meager and V is open with $K \subseteq V$, then there is a countable set of points, say $D(K,V)$, with no point of $D(K,V)$ a limit point of $D(K,V), D(K,V) \subseteq (G \cap V) \setminus K$, and $\overline{D(K,V)} = K \cup D(K,V)$. To see this, let $\{d_1, d_2, \cdots\} \subseteq K$ be dense in K and let for each n, $x_1^{(n)}, \cdots, x_n^{(n)}$ be points of $(G \cap V) \setminus K$ which have distance $< \frac{1}{n}$ from $d_1, \cdots, d_n$, resp. (We are using here that no open ball can be contained in K, and that G is dense in V.) Let $D(K,V) = \{x_i^{(n)} : n \geq 1, i \leq n\}$.

Finally, from our hypothesis, for each nonempty open $U \subseteq X$ there is compact $K_U \subseteq U, K_U \in I \setminus B$.

Let now $G = \bigcap_n W_n, W_n \supseteq W_{n+1}, W_n$ dense open. We will construct for each $s \in 2^{<\mathbb{N}}$, by induction on $|s| = \text{length}(s)$, a compact set K_s and an open set U_s satisfying the following:

(i) $U_s \neq \emptyset, \overline{U}_s \subseteq W_{|s|}, K_s = K_{U_s}$ (so $K_s \in I \setminus B$);

(ii) $\overline{U_{s\hat{}n}} \subseteq U_s, \overline{U_{s\hat{}n}} \cap \overline{\bigcup_{m \neq n} U_{s\hat{}m}} = \emptyset$;

(iii) $K_s \cap \overline{U_{s\hat{}n}} = \emptyset$;

(iv) $\text{diam}(U_{s\hat{}n}) \leq \min\{2^{-|s|}, \frac{1}{n}\}$;

(v) $\overline{\bigcup_s U_{s\hat{}n}} = \bigcup_n \overline{U_{s\hat{}n}} \cup K_s$;

(vi) $K_s \subseteq \overline{\bigcup_n K_{s\hat{}n}}$.

Step 1. $U_\emptyset = W_0, K_\emptyset = K_{U_\emptyset}$.

Step $k+1$. Suppose we have constructed U_s, K_s for $|s| \leq k$, satisfying (i)–(vi). Let for $s \in \mathbb{N}^k$, $D(K_s, U_s) = \{x_{s\hat{}n} : n \in \mathbb{N}\}$ and let $U_{s\hat{}n} \subseteq W_{k+1}$ be a small enough open set containing $x_{s\hat{}n}$ so that $\text{diam}(U_{s\hat{}n}) < \min\{\frac{1}{n}, 2^{-|s|}\}$ and (ii), (iii), (iv) are satisfied. This can be done as no point of $D(K_s, U_s)$ in a limit point of $D(K_s, U_s)$, $D(K_s, U_s) \subseteq (G \cap U_s) \setminus K \subseteq (W_{n+1} \cap U_s) \setminus K_s$, and $\overline{D(K_s, U_s)} = D(K_s, U_s) \cup K_s$. It also follows that $\overline{\bigcup_n U_{s\hat{}n}} = \bigcup_n \overline{U_{s\hat{}n}} \cup K_s$. Also if $K_{s\hat{}n} = K_{U_{s\hat{}n}}$ clearly $K_s \subseteq \overline{\bigcup_n K_{s\hat{}n}}$, as $K_s \subseteq \overline{D(K_s, U_s)}$.

Let $H = \bigcap_n \bigcup_{s \in \mathbb{N}^n} U_s, K = H \cup \bigcup_s K_s$. Clearly H is G_δ and as $\bigcup_{s \in \mathbb{N}^n} U_s \subseteq W_n$, $H \subseteq G$.

Claim. K is closed.

Proof. It is enough to show that if $L = \bigcap_n \overline{\bigcup_{s \in \mathbb{N}^n} U_s}$, then $K = L$. Clearly $H \subseteq \bigcap_n \overline{\bigcup_{s \in \mathbb{N}^n} U_s}$. Since, by (vi), $K_s \subseteq \overline{\bigcup_n K_{s\hat{}n}}$, for all s we have that $K_s \subseteq \overline{\bigcup_n K_{s\hat{}n}}, K_s \subseteq \overline{\bigcup_{n,m} K_{s\hat{}n\hat{}m}}, \cdots$ so if $|s| = k$, $K_s \subseteq \overline{\bigcup_{s \in \mathbb{N}^{k+1}} U_s}, K_s \subseteq \overline{\bigcup_{s \in \mathbb{N}^{k+2}} U_s}, \cdots$, i.e., $K_s \subseteq L$ and so $\bigcup_s K_s \subseteq L$. Thus $K \subseteq \bigcap_n \overline{\bigcup_{s \in \mathbb{N}^n} U_s}$. Let now $x \in \bigcap_n \overline{\bigcup_{s \in \mathbb{N}^n} U_s}$, in order to show that $x \in K$. If $x \in \bigcup_s K_s$ we are done. So assume $x \notin \bigcup_s K_s$, in order to show that $x \in H$. Then, since $x \in \overline{\bigcup_n U_{(n)}}$, we have, by (v), that $x \in \bigcup_n \overline{U_n}$, so $x \in \overline{U_{(n_0)}}$ for some $n_0 \in \mathbb{N}$. Again $x \in \overline{\bigcup_{s \in \mathbb{N}^2} U_s}$, so, as $x \in \overline{U_{(n_0)}}$, by (ii), $x \in \overline{\bigcup_n U_{(n_0,n)}}$, and thus by (v) again $x \in \bigcup_n \overline{U_{(n_0,n)}}$, so $x \in \overline{U_{(n_0,n_1)}}$ for some $n_1 \in \mathbb{N}$, etc. Thus $\forall k(x \in \overline{U_{(n_0,n_1,\cdots,n_k,n_{k+1})}} \subseteq U_{(n_0,n_1\cdots,n_k)})$, so $x \in \bigcap_k \bigcup_{s \in \mathbb{N}^k} U_s$, i.e., $x \in H$.

Claim. $K \notin I$: Otherwise, $K = \bigcup_n K_n,\ K_n \in B$, so by the Baire Category Theorem there is open U_0, and n_0 with $\emptyset \neq U_0 \cap K \subseteq K_{n_0}$, so $\overline{U_0 \cap K} \in B$. Let $x \in U_0 \cap K$. If $x \in H$, then there is unique $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $x \in U_{\alpha|n}$ for each n. Since $\operatorname{diam}(U_{\alpha|n}) \to 0$, there is n with $U_{\alpha|n} \subseteq U_0$, so $K_{\alpha|n} \subseteq U_0$ and thus $K_{\alpha|n} \subseteq U_0 \cap K$, so, as $K_{\alpha|n} \notin B, \overline{U_0 \cap K} \notin B$, a contradiction. If now $x \in K_s$ for some s, then, by (v) and (iv), $K_{s\hat{}n} \subseteq U_0$ for some n, so $K_{s\hat{}n} \subseteq U_0 \cap K$, and again $\overline{U_0 \cap K} \notin B$, a contradiction.

Since $K = H \cup \bigcup_s K_s$ and $K_s \in I$, while $K \notin I$, it follows, by calibration, that $K(K \setminus \bigcup_s K_s) \not\subseteq I$, so as $K \setminus \bigcup_s K_s \subseteq H, K(H) \not\subseteq I$, i.e., there is $K \subseteq H, K \in K(X), K \notin I$. But $H \subseteq G$, so $\exists K \in K(X), K \subseteq H, K \notin I$ and the proof is complete. $\dashv$

The following is an important application of the Basis Theorem.

23.3 The Covering Theorem (Debs-Saint Raymond). *Let I be a σ-ideal of closed sets in X. Assume*

(i) I is calibrated,

(ii) I admits a basis B such that for any $L \in K(X) \setminus I$, there is $K \in K(L), K \in I \setminus B$.

Then for any $\mathbf{\Sigma}^1_1$ set $A \subseteq X$, if $K(A) \subseteq I$, then there is a sequence $\{K_n\} \subseteq I$ with $A \subseteq \bigcup_n K_n$.

Proof. Assume $A \subseteq X$ is $\mathbf{\Sigma}^1_1$, and A cannot be covered by countably many sets in I. We will show that there is $K \in K(X), K \notin I$ with $K \subseteq A$.

By 17.1, let $G \subseteq X \times \mathcal{C}$ be G_δ such that $A = \operatorname{proj}_X[G]$. Let $G' = G \setminus \bigcup\{V$ open in $X \times \mathcal{C}$: $\operatorname{proj}_X[V \cap G]$ can be covered by countably many sets from $I\}$. Since A cannot be covered by countably many sets from I, $G' \neq \emptyset$. Let $F = \overline{G'}$, so that F is compact. We define the following σ-ideal $J \subseteq K(F)$:

$$K \in J \Leftrightarrow \operatorname{proj}_X[K] \in I.$$

We claim that J satisfies the hypothesis of the Basis Theorem 23.2, i.e., is calibrated and has a non-trivial basis. It will follow then (as G' is dense in G_δ in F) that there is compact $L \subseteq F$ with $L \subseteq G'$ and $L \notin J$, i.e., $\operatorname{proj}_X[L] = K \in K(A) \setminus I$, so we are done.

J is calibrated: Let $K \in K(F), \{K_n\} \subseteq J, K(K \setminus \bigcup_n K_n) \subseteq J$. Consider $\operatorname{proj}_X[K] \setminus \bigcup_n \operatorname{proj}_X[K_n]$. We have $\operatorname{proj}_X[K_n] \in I$. If $K \notin J$, towards a contradiction, then $\operatorname{proj}_X[K] \notin I$, so by the calibration of I, $\operatorname{proj}_X[K] \setminus \bigcup_n \operatorname{proj}_X[K_n]$ contains a compact set L with $L \notin I$. Then $K \cap (L \times \mathcal{C}) \subseteq K \setminus \bigcup_n K_n$ is not in J (as $\operatorname{proj}_X[K \cap (L \times \mathcal{C})] = L$), a contradiction.

J has a non-trivial basis: Let $D = \{K \in K(F)$: $\operatorname{proj}_X[K] \in B\}$. Then D is a basis for J. If V is an open set intersecting F, then, by the definition of G', $\operatorname{proj}_X[V \cap G']$ cannot be covered by countably many sets in I (otherwise $\operatorname{proj}_X[V \cap G]$ can be so covered, so $V \cap G' = \emptyset$, contradicting the density of G' in F). So $L = \overline{\operatorname{proj}_X[V \cap F]} = \overline{\operatorname{proj}_X[V \cap G']} \notin I$. Then there is $K \subseteq L, K \in I \setminus B$. Now $L = \overline{\operatorname{proj}_X[V \cap F]} = \operatorname{proj}_X[\overline{V \cap F}]$ (by compactness), so

by looking at $(K \times \mathcal{C}) \cap [\overline{V \cap F}]$ we conclude that $J \cap K(\overline{V \cap F}) \neq D \cap K(\overline{V \cap F})$. Since this is true for *all* open V that intersect F, it follows that $J \cap K(U) \neq D \cap K(U)$ for all non-$\emptyset$ U open relative to F and the proof is complete. $\dashv$

The following corollary gives a definability context under which 23.3 can be applied.

23.4. Corollary. *Let I be a σ-ideal of closed sets in X. Assume*

(i) I is calibrated;

(ii) if $L \in K(X) \setminus I$, then $I \cap K(L)$ is not Borel;

(iii) I admits a Borel basis.

Then for any $\mathbf{\Sigma}^1_1$ set $A \subseteq X$, if $K(A) \subseteq I$, then there is a sequence $\{K_n\} \subseteq I$ with $A \subseteq \bigcup_n K_n$.

§24. Non-existence of Borel bases for U.

We will now apply the methods of §23 to the σ-ideal U of closed sets of uniqueness.

Debs and Saint Raymond used a deep result from harmonic analysis due to Körner (the existence of so-called Helson sets of multiplicity), to show that there is a closed set E such that for every open W with $W \cap E \neq \emptyset, \overline{W \cap E} \notin U$ and a G_δ set $G \subseteq E$ which is dense in E such that $K(G) \subseteq U$. This is one of the ingredients needed to apply 23.2. The other two ingredients are

(i) If $E \in M = K(\mathbb{T}) \setminus U$, then $U \cap K(E)$ is not Borel.

This is a local version of the Kaufman-Solovay Theorem 18.1 and has been proved independently by Debs-Saint Raymond, Kaufman and Kechris-Louveau.

(ii) U is calibrated.

This was proved independently by Debs-Saint Raymond and Kechris-Louveau. The proofs of all these results are given in Kechris-Louveau [1989].

Putting all them together we have:

24.1. Theorem (Debs-Saint Raymond). *The σ-ideal U of closed sets of uniqueness has no Borel basis.*

Proof. Assume U had a Borel basis $B \subseteq U$. Let $E \in K(\mathbb{T})$ be such that for every open W with $W \cap E \neq \emptyset$ we have $\overline{W \cap E} \notin U$, but there is a G_δ set $G \subseteq E$, dense in E, with $K(G) \subseteq U$. Now consider

$$I = U \cap K(E).$$

It is a $\mathbf{\Pi}^1_1$ σ-ideal of closed subsets of $K(E)$. It is (i) calibrated, (ii) for every $L \in K(E), L \notin I, I \cap K(L)$ is not Borel and (iii) I admits a Borel basis, namely $B \cap K(E)$. So by 23.4 applied to $A = G$ we must have that there is a sequence $K_n \in I$ with $G \subseteq \bigcup_n K_n$. But each K_n is meager in E, since otherwise there would be W open with $W \cap E \neq \emptyset$ and $W \cap E \subseteq K_n$, so $\overline{W \cap E} \subseteq K_n$, a contradiction since $\overline{W \cap E} \in M$ and $K_n \in U$. So G is meager, a contradiction. $\dashv$

This result, as we explained earlier, has very strong implications concerning the Characterization Problem for U-sets. One cannot characterize U-sets as countable unions of any reasonably explicitly characterizable subclass, e.g., H-sets, $H^{(n)}$-sets, etc. Or, one can use this as an existence theorem: Given any reasonable explicitly characterizable sublcass of U-sets, say B, there is a closed set $E \in U$ which is not a countable union of sets in B. Thus this gives a new proof that for each n there are U-sets which are not countable unions of $H^{(n)}$-sets (a result originally due to Piatetski-Shapiro) or that there are U-sets which are not countable unions of $H^{(n)}$-sets for varying n (a new result), etc.

§25. Existence of a Borel basis for U_0.

Recall that U_0 is the class of all closed sets of extended uniqueness, i.e., $K \in U_0$ if for every measure $\mu \in M(\mathbb{T})$, if $\sum \hat{\mu}(n)e^{inx} = 0$ off K, then $\hat{\mu}(n) = 0, \forall n \in \mathbb{Z}$ (i.e., $\mu = 0$). By 21.3, this is equivalent to saying that there is no $\mu \in M(\mathbb{T}), \mu \neq 0$, with $\hat{\mu}(n) \to 0$ which is supported by K. By using a bit of additional measure theory we can see that in this characterization we can restrict ourselves to probability Borel measures, i.e., $\mu \in P(\mathbb{T})$. To see this, it is enough to check that if $\mu \neq 0, \mu \in M(\mathbb{T}), \hat{\mu}(n) \to 0$ and μ is supported by K, then there is $\mu \in P(\mathbb{T})$ with $\hat{\mu}(n) \to 0$ also supported by K.

For this let $|\mu|$ be the so-called *total variation* of μ. This is the finite positive Borel measure defined by

$$|\mu|(A) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|,$$

where the sup varies over all Borel partitions of A. Thus

$$||\mu||_M = |\mu|(\mathbb{T}).$$

Moreover, by the Radon-Nikodym Theorem, there is a Borel function $h : \mathbb{T} \to \mathbb{C}$ with $|h| = 1$ such that

$$\mu(E) = \int hd|\mu|,$$

and so

$$\int fd\mu = \int fhd|\mu|$$

for any bounded Borel f. Thus, in particular,

$$\int fh^{-1}d\mu = \int fd|\mu|.$$

Note also that $\left| \int fd\mu \right| \leq \int |f|d|\mu|$.

It follows that $|\mu|$ is supported by K. It is thus enough to show that $\widehat{|\mu|}(n) \to 0$. (Although $|\mu|$ might not be a probability measure, $\nu = |\mu|/|\mu|(\mathbb{T})$ will be, it will have the same support as $|\mu|$, and $\hat{\nu}(n) \to 0$ as well.)

Fix $\epsilon > 0$. As $h^{-1} \in L_1(|\mu|)$ (since $\int |h^{-1}| d|\mu| = \int d|\mu| = |\mu|(\mathbb{T}) < \infty$), there is a trigonometric polynomial $P(x) = \sum_{k=-N}^{N} c_n e^{ikx}$ with

$$||h^{-1} - P||_{L^1(|\mu|)} = \int \Big| h^{-1} - P \Big| d|\mu| < \epsilon.$$

Then

$$\int e^{-inx} d|\mu|(x) = \int e^{-inx} h^{-1} d\mu(x)$$

$$= \int e^{-inx} P(x) d\mu(x) -$$

$$\int e^{-inx}(P(x) - h^{-1}(x)) d\mu(x)$$

$$= \sum_{k=-N}^{N} c_k \hat{\mu}(n-k) -$$

$$\int e^{-inx}(P(x) - h^{-1}(x)) d\mu(x).$$

But

$$\Big| \int e^{-inx}(P(x) - h^{-1}(x)) d\mu(x) \Big| \leq \int \Big| h^{-1} - P \Big| d|\mu| < \epsilon$$

and, since $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, it follows that there is N_ϵ such that for $|n| \geq N_\epsilon$,

$$\Big| \sum_{k=-N}^{N} c_n \hat{\mu}(n-k) \Big| < \epsilon,$$

so for $|n| \geq N_\epsilon$, $\Big| \int e^{inx} d|\mu|(x) \Big| < 2\epsilon$ and thus $\widehat{|\mu|}(n) \to 0$ as $|n| \to \infty$.

Thus we have (recalling that $R = \{\mu \in P(\mathbb{T}) : \hat{\mu}(n) \to 0\}$):

25.1. Proposition. *For any $K \in K(\mathbb{T})$,*

$$K \in U_0 \Leftrightarrow \forall \mu \in P(\mathbb{T})(K \textit{ supports } \mu \Rightarrow \mu \notin R).$$

For $\mu \in P(\mathbb{T})$ let

$$R(\mu) = \overline{\lim} |\hat{\mu}(n)|.$$

Thus $0 \leq R(\mu) \leq 1$ and

$$\mu \in R \Leftrightarrow R(\mu) = 0.$$

Put

$$U_0' = \{K \in K(\mathbb{T}) : \exists \epsilon > 0 \forall \mu \in P(\mathbb{T})(K \text{ supports } \mu \Rightarrow R(\mu) \geq \epsilon).$$

Thus clearly $U_0' \subseteq U_0$. (Note that $U_0 = \{K \in K(\mathbb{T}) : \forall \mu \in P(\mathbb{T})\ (K \text{ supports } \mu \Rightarrow R(\mu) > 0)\}$.)

For example, it turns out that the Cantor set $E_{1/3}$ is in U_0'. In fact we have:

25.2. Proposition. *Every closed H-set is in U_0'.*

Proof. Let E be a closed H-set and let $0 < n_0 < n_1 < \cdots$ be a sequence and I an open interval with $n_i x \notin I$ for any $x \in E$. Let $\varphi \in A$ be a function supported by some closed interval contained in I and $\hat{\varphi}(0) = 1$. Put $f_k(x) = \varphi(n_k x)$. Then $f_k(x) = 0$ if $x \in E$, so for any $\mu \in P(E)$, $\int f_k d\mu = \langle f_k, \mu\rangle = \sum \hat{f}_k(n)\hat{\mu}(-n) = 0$. Note that $\hat{f}_k(0) = 0$ and $\hat{f}_k(n) \to 0$ as $k \to \infty$, for any $n \neq 0$.

Take $\epsilon > 0$ such that $\epsilon \cdot ||\varphi||_A < 1$. We claim that if $\mu \in P(E)$, then $R(\mu) \geq \epsilon$. Otherwise let $\mu \in P(E)$ be such that $R(\mu) < \epsilon$. Then fix $N \in \mathbb{N}$ so that $|\hat{\mu}(n)| < \epsilon$ for any $|n| > N$. We have

$$0 = \sum \hat{f}_k(n)\hat{\mu}(-n)$$

$$= 1 + \sum_{\substack{n \neq 0 \\ n=-N}}^{N} \hat{f}_k(n)\hat{\mu}(-n) + \sum_{|n|>N} \hat{f}_k(n)\hat{\mu}(-n),$$

thus

$$1 \leq \left| \sum_{\substack{n \neq 0 \\ n=-N}}^{N} \hat{f}_k(n) \right| + \sum_{|n|>N} |\hat{f}_k(n)| \cdot \epsilon.$$

The first summand can be made arbitrarily small by letting $k \to \infty$ and the second is bounded by $||\varphi||_A \cdot \epsilon < 1$, so we have a contradiction. $\dashv$

25.3. Proposition. *U_0' is Borel and hereditary.*

Proof. We have for $K \in K(\mathbb{T})$

$$K \notin U_0' \Leftrightarrow \forall \epsilon \exists \mu \in P(\mathbb{T})(K \text{ supports } \mu \ \&\ R(\mu) < \epsilon)$$

$$\Leftrightarrow \forall \epsilon \exists \mu \in P(\mathbb{T})(K \text{ supports } \mu \ \&\ \exists n \forall m(|m| > n \Rightarrow |\hat{\mu}(m)| \leq \epsilon)).$$

Now $P = \{(K, \mu) : K \text{ supports } \mu \ \&\ \exists n \forall m(|m| > n \Rightarrow |\hat{\mu}(m)| \leq \epsilon)\}$ is F_σ in $K(\mathbb{T}) \times P(\mathbb{T})$, so $K(\mathbb{T}) \setminus U_0'$ is $\mathbf{\Pi}^0_3$, and thus U_0' is $\mathbf{\Sigma}^0_3$.

That U'_0 in hereditary is obvious. $\dashv$

25.4. Theorem (Kechris-Louveau). *U'_0 is a basis for U_0, so U_0 admits a Borel basis.*

Proof. Let $E \in K(\mathbb{T}), E \notin (U'_0)_\sigma$. We will show that $E \notin U_0$. Since $K \notin (U'_0)_\sigma$, by 23.1, there is $\emptyset \neq F \subseteq E, F \in K(\mathbb{T})$ which is U'_0-perfect, i.e., for any open V.

$$F \cap V \neq \emptyset \Rightarrow \overline{F \cap V} \notin U'_0,$$

i.e., $F \cap V \neq \emptyset \Rightarrow \forall \epsilon \exists \mu \in P(\mathbb{T})$ ($\overline{F \cap V}$ supports μ and $R(\mu) < \epsilon$). We will find $\mu \in R$ supported by F, so $F \notin U_0$ and thus $E \notin U_0$.

Consider first $P(F)$, the set of probability measures supported by F. This is a closed subset of $P(\mathbb{T})$ (always equipped with the weak*-topology).

Let

$$R_\epsilon = \{\mu \in P(\mathbb{T}) : R(\mu) < \epsilon\}$$

and

$$R_\epsilon(F) = P(F) \cap R_\epsilon.$$

We claim that $R_\epsilon(F)$ is dense in $P(F)$. Since $R_\epsilon(F)$ is convex, it suffices (by 10.14) to show that every Dirac measure $\delta_x, x \in F$, is in the closure of $R_\epsilon(F)$. Let (by 10.4) V_n be a sequence of nbhds of x with $\operatorname{diam}(V_n) \to 0$. Then $\overline{V_n \cap F} \notin U'_0$, so there is $\mu_n \in P(\overline{V_n \cap F})$ with $\mu_n \in R_\epsilon(F)$. But $\mu_n \longrightarrow^{w^*} \delta_x$ (see §11) and so we are done.

We will now construct, by induction on n, a sequence $\mu_n \in P(F)$, and $0 < N_0 < N_1 < N_2 < \cdots < N_n < \cdots$ such that for each n:

$$(*) \qquad i \leq n, N_i \leq |k| < N_{i+1} \Rightarrow |\hat{\mu}_n(k)| < 2^{-i-1}.$$

Then, by the compactness of $P(F)$, let μ be a w^*-limit of a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$. We clearly have (as $\hat{\mu}_{n_j}(k) \to \hat{\mu}(k)$ for every k) that $|\hat{\mu}(k)| \leq 2^{-i-1}$ if $|k| \geq N_i$, so $\mu \in R$ and $\mu \in P(F)$, thus we are done.

To construct $\{\mu_n\}, \{N_n\}$ we will actually choose μ_n to satisfy $(*)$ for $i \leq n-1$ and

$$(**) \qquad \forall |k| \geq N_n (|\hat{\mu}_n(k)| < 2^{-n-2}).$$

(Then $(*)$ will be satisfied for $i = n$ no matter what N_{n+1} is).

$n = 0$. Find $\mu_0 \in P(F)$ so that $\mu_0 \in R_{2^{-2}}(F)$ and then choose N_0 so that $|\hat{\mu}_0(k)| < 2^{-2}$ for $|k| \geq N_0$.

$n \to n+1$. Suppose $\mu_0, \cdots \mu_n, N_0, \cdots, N_n$ have been defined satisfying $(*)$ for $i \leq n-1$ and $(**)$. Let $m \geq N_n$. Then there is $\mu^m \in P(F)$ and $\varphi(m) > m$ such that

(i) μ^m satisfies $(*)$ for $i \leq n-1$,

(ii) $|\mu^m(k)| < 2^{-n-2}$ for $N_n \leq |k| < m$,

(iii) $|\mu^m(k)| < 2^{-n-3}$ for $|k| \geq \varphi(m)$.

This is because μ_n satisfies (i), (ii) and so, by the density of $R_{2^{-n-3}}(F)$, there is $\mu^m \in R_{2^{-n-3}}(F)$ satisfying (i), (ii). Then choose $\varphi(m)$ to make (iii) true.

Now define a sequence $\nu_j \in P(F)$ and m_j by

$$\nu_0 = \mu^{N_n}, m_0 = \varphi(N_n),$$

$$\nu_{j+1} = \mu^{m_j}, m_{j+1} = \varphi(m_j).$$

Let for each k

$$\theta_k = \frac{1}{k+1}\sum_{j=0}^{k} \nu_j.$$

Then θ_k satisfies $(*)$ for $i \le n-1$ and $|\hat{\theta}_k(m)| < 2^{-n-3}$ for $|m| \ge m_k$. If

$$N_n \le |m| < m_k,$$

there is at most one j, namely the one such that $m_j \le |m| < m_{j+1}$ for which $|\nu_j(m)| \ge 2^{-n-2}$. So (as always $|\widehat{\nu_j(p)}| \le 1$),

$$|\hat{\theta}_k(m)| \le \frac{k \cdot 2^{-n-2}+1}{k+1}.$$

Choose then k large enough so that $\frac{k\cdot 2^{-n-2}+1}{k+1} < 2^{-n-1}$. Put $\mu_{n+1} = \theta_k, N_{n+1} = m_k$. Then clearly μ_{n+1} satisfies $(*)$ for $i \le n-1$ (as each ν_j does). Also for $N_n \le |k| < N_{n+1}, |\hat{\mu}_{n+1}(k)| < 2^{-n-1}$, so μ_{n+1} satisfies $(*)$ for $i \le n$, and finally it clearly satisfies $(**)$, i.e., $\forall |k| \ge N_{n+1}(|\hat{\mu}_{n+1}(k)| < 2^{-n-3})$. ⊣

We are now in a position to apply 23.3. It is clear that U_0 is calibrated. Kaufman had showed that for every $L \in M_0$, there is $K \subseteq L, K \in U_0 \setminus U_0'$ (see also §27 below). So by 23.3 we have:

25.5. Theorem (Debs-Saint Raymond). *For each $\mathbf{\Sigma}^1_1$ set $A \subseteq \mathbb{T}$, if $A \in \mathcal{U}_0$ there is a sequence $K_n \in U_0$ with $A \subseteq \bigcup_n K_n$. In particular, A is of the first category.*

Proof. Recall that by definition

$$A \in \mathcal{U}_0 \Leftrightarrow \forall \mu \in M(\mathbb{T})(\sum \hat{\mu}(n)e^{inx} = 0 \text{ off } E \Rightarrow \hat{\mu}(n) = 0, \forall n \in \mathbb{Z}).$$

Then we must have for every Rajchman measure $\mu \in R, \mu(A) = 0$. (Otherwise, $\mu(A) > 0$, since every $\mathbf{\Sigma}^1_1$ set is μ-measurable (see Kechris [1995, 29.7]. Then there is $F \in K(\mathbb{T})$ with $\mu(F) > 0$. Let $\nu = (\mu|F)/\mu(F)$. Then by 9.3, $\hat{\nu}(n) \to 0$ and so, by 7.6, $\sum \hat{\nu}(n)e^{inx} = 0, \forall x \notin F$, and so $\forall x \notin A$, a contradiction.) So by 23.3 there are $K_n \in U_0$, with $A \subseteq \bigcup_n K_n$. ⊣

25.6. Corollary. *Let $A \subseteq \mathbb{T}$ have the BP and be in $\mathcal{U}_0$. Then A is of the first category.*

Proof. If A is not of the first category, then A is comeager in some open set U, so A contains a G_δ set G which is dense in U. Then G is obviously $\mathbf{\Sigma}^1_1$ and in $\mathcal{U}_0$ but not of the first category, a contradiction. $\dashv$

This was the original solution of the Category Problem 9.1 (which in fact established the stronger version about sets of extended uniqueness with the BP being of the first category). In §11 we gave a different proof based on Baire Category methods. Such a technique can be also used to give a proof of 25.5 as well.

In conclusion we can summarize as follows the key structural and definability properties of U, U_0:

For I either U or U_0 we have

(i) I is a calibrated σ-ideal,

(ii) I is $\mathbf{\Pi}^1_1$, but for any $F \in K(\mathbb{T}), F \notin I, I \cap K(F)$ is not Borel.

These are properties that both U and U_0 share. However they differ in one important respect.

(iii) U_0 has a Borel basis, but U does not have a Borel basis.

It should be noted that the three properties (i)–(iii) encapsulate a large part of the theory of sets of uniqueness (and extended uniqueness). For example, they imply Menshov's Theorem (existence of null closed sets of multiplicity); the Debs-Saint Raymond Theorem (sets of extended uniqueness with the BP are of the first category), which in turn has several consequences, like Lyons' theorem that there are Rajchman measures supported by the non-normal numbers; Piatetski-Shapiro's Theorem that $U \neq U_0$, etc.

§26. Co-analytic ranks.

The last topic in these lectures will bring us back full circle to the first method we introduced here, ordinals and transfinite induction. Ordinals play a crucial role in classical as well as modern descriptive set theory, through various concepts of *rank.* For our purposes here, the crucial concept is that of a *co-analytic* or $\mathbf{\Pi}^1_1$-*rank.* A general reference for descriptive set theoretic results used in this section is Kechris [1995].

A *rank* on a set A is simply a function $\varphi : A \to$ Ordinals, assigning to each element of A an ordinal number. It is a fundamental property of all co-analytic sets that they admit ranks $\varphi : A \to \omega_1$ (= the set of all countable ordinals) with very nice definability properties. Roughly speaking, such φ exist for which the initial segments

$$\{x \in A : \varphi(x) \leq \varphi(y)\},$$

for $y \in A$, are "uniformly" $\mathbf{\Delta}^1_1$. Let me be more precise.

Definition. Let X be a Polish space and $A \subseteq X$ a $\mathbf{\Pi}^1_1$-set. A $\mathbf{\Pi}^1_1$-*rank* on A is a map $\varphi : A \to \omega_1$ which has the following property:

There are $P, S \subseteq X^2$, in $\mathbf{\Pi}^1_1, \mathbf{\Sigma}^1_1$, resp., such that

$$y \in A \Rightarrow [x \in A \ \& \ \varphi(x) \leq \varphi(y) \Leftrightarrow (x,y) \in P \Leftrightarrow (x,y) \in S].$$

Thus for $y \in A$,

$$\{x \in A : \varphi(x) \leq \varphi(y)\} = P^y = S^y$$

(where $R^y = \{x : (x,y) \in R\}$), so that the initial segment determined by y is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, i.e., $\mathbf{\Delta}^1_1$, but in fact in a uniform way.

It is a basic fact of the theory of $\mathbf{\Pi}^1_1$ sets that they admit $\mathbf{\Pi}^1_1$-ranks.

26.1. Theorem. *Every $\mathbf{\Pi}^1_1$ set A admits a $\mathbf{\Pi}^1_1$-rank.*

Note that if $\varphi : A \to \omega_1$ is a $\mathbf{\Pi}^1_1$-rank on the $\mathbf{\Pi}^1_1$ set A, and for each countable ordinal ξ we let

$$A_\xi = \{x \in A : \varphi(x) \leq \xi\},$$

then A_ξ is Borel. (*Proof.* Recall that $\mathbf{\Delta}^1_1$ = Borel. This is now easily proved by induction on ξ. If $\xi = 0$, then either $A_0 = \emptyset$, or else if $y \in A_0$, clearly $A_0 = \{x \in A : \varphi(x) \leq \varphi(y)\}$ is Borel. Assume it holds for all $\xi < \eta$, and consider A_η. If there is no $y \in A$ with $\varphi(y) = \eta$, clearly $A_\eta = \bigcup_{\xi<\eta} A_\xi$ is Borel, as this is a countable union of Borel sets. If on the other hand there is $y \in A$ with $\varphi(y) = \eta$, then clearly $A_\eta = \{x \in A : \varphi(x) \leq \varphi(y)\}$, so A is again Borel.)

Since

$$A = \bigcup_{\xi<\omega_1} A_\xi,$$

this gives a decomposition of A as a union of ω_1 many Borel sets. So, although A may not be Borel, it can be "approximated" by Borel sets, in some sense.

An important application of the concept of $\mathbf{\Pi}^1_1$-rank is the following:

26.2 Boundedness Theorem. *Let A be a $\mathbf{\Pi}^1_1$ set and $\varphi : A \to \omega_1$ a $\mathbf{\Pi}^1_1$-rank. If $B \subseteq A$ is $\mathbf{\Sigma}^1_1$, then there is a countable ordinal ξ such that*

$$\varphi(x) \leq \xi$$

for all $x \in B$.

This result suggests the following technique for establishing the non-Borelness of a given $\mathbf{\Pi}^1_1$ set, called the *rank method:* Given a $\mathbf{\Pi}^1_1$ set A, for which we want to show that it is not Borel (equivalently: not analytic), find a $\mathbf{\Pi}^1_1$-rank $\varphi : A \to \omega_1$ which is *unbounded,* i.e., for each countable ordinal ξ there is some $x \in A$ with $\varphi(x) > \xi$.

For example, one can use this method to give another proof that the set $K_\omega(X)$ of countable closed subsets of an uncountable compact metric space X is not Borel (see 22.4). One lets $\varphi(K)$ = the Cantor-Bendixson rank of K. Then it turns out that φ is a $\mathbf{\Pi}^1_1$-rank. Since, as in 5.6, one can see that φ is unbounded, this shows that $K_\omega(X)$ is not Borel.

For this and other reasons it is important, when studying the descriptive properties of a given $\mathbf{\Pi}_1^1$ set, like U or U_0, to find canonical $\mathbf{\Pi}_1^1$-ranks that reflect the structural properties of the sets (and don't just come from applying the abstract Theorem 26.1). There is indeed such a canonical $\mathbf{\Pi}_1^1$-rank for U, called the *Piatetski-Shapiro rank* (see Kechris-Louveau [1989]), but I will not discuss this here. There is also a canonical rank for U_0, which I will now discuss, since it has a particularly simple description using the basis U_0' for U_0 and the generalized Cantor-Bendixson procedure described in 23.1.

Let X be a compact metric space, and $B \subseteq K(X)$ a hereditary Borel set. Let $I = B_\sigma$ be the σ-ideal generated by B. By 23.1, I is $\mathbf{\Pi}_1^1$. Define, as in 23.1 again, the following rank on I:

$$\varphi(K) = rk_B(K).$$

Let us call this the *B-rank* of I. The following can be then proved.

26.3. Theorem. *For any hereditary Borel $B \subseteq K(X)$, the B-rank on $I = B_\sigma$ is a $\mathbf{\Pi}_1^1$-rank.*

This is of course a generalization of the corresponding fact for the Cantor-Bendixson rank, which we used in the example above. It also shows that the U_0'-rank on U_0 is a $\mathbf{\Pi}_1^1$-rank on U_0.

One then can use the rank method to establish that $I = B_\sigma$ is not Borel: It is enough for that to show that for each countable ordinal ξ there is $K \in B_\sigma$ with $rk_B(K) > \xi$.

For the following exercise note that if B is an ideal (i.e., it is also closed under finite unions), then $rk_B(K) \leq 1 \Leftrightarrow K \in B$.

26.4. Exercise. Assume $B \subseteq K(X)$ is a Borel ideal consisting of nowhere dense sets and $J \subseteq B_\sigma$ is a σ-ideal such that for every open non-$\emptyset$ $V \subseteq X$ there is $K \in J \setminus B, K \subseteq V$. Show that for every countable ordinal ξ, there is a $K \in J$ with $rk_B(K) > \xi$. Conclude that J is not analytic. [*Hint.* Show by transfinite induction that for every countable ordinal ξ, and for every open non-$\emptyset$ set $V \subseteq X$, there is $K \in J, K \subseteq V$ with $rk_B(K) > \xi + 1$.)

We will apply this method to show that U_0 is not Borel. In fact, by applying appropriately 26.4 we will see a much stronger property of U_0. In a sense, U_0 is "hereditarily" non Borel.

§27. A hereditary property of U_0.

I will prove here the following:

27.1. Theorem (Kechris). *There is no simple (i.e., Borel, or, equivalently, G_δ) σ-ideal I such that*

$$K_\omega(\mathbb{T}) \subseteq I \subseteq U_0.$$

Before giving the proof, I want to make some comments.

(i) Since $K_\omega(\mathbb{T}) \subseteq U \subseteq U_0$, this shows that U, U_0 (as well as $K_\omega(\mathbb{T})$) are not Borel, so it gives another proof of these results.

(ii) The proof also shows that the hypothesis can be relaxed to the following: For any $E \in K(\mathbb{T})$ which is U_0-perfect, there is no Borel ideal with $K_\omega(E) \subseteq I \subseteq U_0$. In particular, this implies that for any $E \in K(\mathbb{T}), E \notin U_0$, $U_0 \cap K(E)$ is not Borel (which was one of the ingredients needed to apply 23.4 to U_0).

(iii) This result has also implications concerning characterizations of subclasses of closed uniqueness sets. For example, A. Olevskii proposed, in a private conversation, a specific explicit characterization of the following class of uniqueness sets:

$$U_{\text{diff}} = \{E \in U : \text{ for every diffeomorphism } h \text{ of } \mathbb{T}\ h[E] \in U\}.$$

Since clearly U_{diff} is a σ-ideal and

$$K_\omega(\mathbb{T}) \subseteq U_{\text{diff}} \subseteq U_0,$$

it follows that U_{diff} cannot be Borel. However, Olevskii's proposed characterization would easily lead to a Borel definition of U_{diff}, which shows that this proposed characterization does not work.

(iv) Kechris-Louveau-Woodin have shown that any $\mathbf{\Sigma}^1_1$ σ-ideal of closed sets is actually simple (i.e., Borel). So 27.1 also implies that there is no $\mathbf{\Sigma}^1_1$ ideal between $K_\omega(\mathbb{T})$ and U_0.

Proof of 27.1. We will apply 26.4 to $X = \mathbb{T}, B = U_0', J = I$. First we will verify that U_0' is an ideal, i.e., is closed under finite unions. This is due to Lyons and Host-Parreau. We need a couple of lemmas first.

Lemma 1. *Let μ, ν be finite, positive Borel measures on $\mathbb{T}$ so that $\nu(\mathbb{T}) \leq 1$ and $\nu \leq \mu$ (i.e., $\nu(A) \leq \mu(A)$ for all Borel A or, equivalently, $\int f d\nu \leq f d\mu$ for all Borel $f \geq 0$). Let $n_1, \cdots, n_k \in \mathbb{Z}$ and let $w_j \in \mathbb{C}$ be such that $|\hat{\nu}(n_j)| = w_j \hat{\nu}(n_j)$ (we agree that $w_j = 1$, if $\hat{\nu}(n_j) = 0$). Then*

$$\left(\frac{1}{k}\sum_{j=1}^{k} |\hat{\nu}(n_j)|\right)^2 \leq \frac{\hat{\mu}(0)}{k} + \frac{2}{k^2} \sum_{1 \leq a < b \leq k} \text{Re } w_b\ \bar{w}_a\ \hat{\mu}(n_b - n_a).$$

Proof. We have

$$\left(\frac{1}{k}\sum_{j} |\hat{\nu}(n_j)|\right)^2 = \frac{1}{k^2}\left(\sum_{j} w_j \int e^{-in_j x} d\nu\right)^2$$

$$\leq \frac{1}{k^2}\left(\int \left|\sum_{j} w_a e^{-in_j x}\right| d\nu\right)^2.$$

Letting $f(x) = \sum\limits_{j} w_a e^{-in_j x}$ we have, by the Cauchy-Schwartz inequality, that

$$\left(\int |f| d\nu\right)^2 \leq \left(\int |f|^2 d\nu\right) \cdot \left(\int 1 d\nu\right) \leq \int |f|^2 d\nu,$$

as $\nu(\mathbb{T}) \leq 1$. So, since $\nu \leq \mu$,

$$\Big(\frac{1}{k}\sum_j |\hat{\nu}(n_j)|\Big)^2 \leq \frac{1}{k^2}\int\Big(\sum_a w_a e^{-in_a x}\Big)\overline{\Big(\sum_b w_b e^{-in_b x}\Big)}d\mu$$

$$= \frac{\hat{\mu}(0)}{k} + \frac{2}{k^2}\sum_{a<b} \mathrm{Re}\ w_b \bar{w}_a \hat{\mu}(n_b - n_a).$$

⊣

Lemma 2. *If ν, μ are as in Lemma 1, then $R(\nu) \leq R(\mu)^{1/2}$.*

Proof. Suppose that for some $0 < n_1 < n_2 < \cdots$ we have $|\hat{\nu}(n_j)| \geq t$. (Similarly we can deal with the case $0 > n_1 > n_2 \cdots$.) Using the preceding lemma for $n_1, \cdots, n_k$ we get

$$\frac{2}{k^2}\sum_{1\leq a<b\leq k} |\hat{\mu}(n_b - n_a)| \geq t^2 - \frac{\hat{\mu}(0)}{k},$$

so

$$R(\mu) \geq \overline{\lim_k}\Big(\frac{2}{k^2}\sum_{1\leq a<b\leq k} |\hat{\mu}(n_b - n_a)|\Big) \geq t^2.$$

⊣

So let us assume that $E, F \in U_0'$, in order to show that $E \cup F \in U_0'$. Pick ϵ such that $1 > \epsilon > 0$ and $R(\mu) \geq \epsilon$ for any $\mu \in P(E) \cup P(F)$. Now consider any $\mu \in P(E \cup F)$.

If $\mu(E) = 0$ it follows that $\mu \in P(F)$, thus $R(\mu) \geq \epsilon$. So we can assume that $\mu(E) > 0, \mu(F) > 0$. Let $\mu_1 = \mu|E, \mu_2 = \mu|F$, so that $\mu_i \leq \mu$ and $\mu_i(\mathbb{T}) \leq 1$. By Lemma 2,

$$R(\mu)^{1/2} \geq R(\mu_1) \geq \epsilon \cdot \mu(E)$$

$$R(\mu)^{1/2} \geq R(\mu_2) \geq \epsilon \cdot \mu(F).$$

By adding, we get that

$$R(\mu)^{1/2} \geq \epsilon/2,$$

so $R(\mu) \geq \epsilon^2/4$. Thus, in any case, $R(\mu) \geq \epsilon^2/4$ for any $\mu \in P(E \cup F)$ and we are done.⊣

It is clear that U_0' consists of nowhere dense sets, so the final (and main) claim, required to prove the theorem, is to verify that for every open non-$\emptyset$ $V \subseteq \mathbb{T}$, there is $K \subseteq V, K \in I \setminus U_0'$.

The main lemma is the following, where for any probability Borel measure μ, $\mathrm{supp}(\mu)$ is the smallest closed set supporting μ, i.e., $\mathrm{supp}(\mu) = \mathbb{T} \setminus \bigcup\{V : V \text{ open and } \mu(V) = 0\}$.

Lemma 3. *Let $E = [a, b]$ be a closed interval, $a < b$ and let $\mu = (\lambda|E)/\lambda(E)$ (so that $\mu \in R$ and $\mathrm{supp}(\mu) = E$). Let $J \subseteq K(E)$ be G_δ, hereditary and assume it contains all finite* *subsets of* E. *Then for any* $N > 0, \epsilon > 0$ *there is* $\nu \in P(E)$ *with supp*$(\nu) = E_0 \cup \cdots \cup E_{N-1}$, *where* $E_n \in J$, *and*

$$\sup_{j \in \mathbb{Z}} |\hat{\mu}(j) - \hat{\nu}(j)| \leq \frac{(1+\epsilon)}{N}.$$

Granting this, the proof of the above claim can be completed as follows:

Fix an open non-$\emptyset$ interval $V \subseteq \mathbb{T}$ and let $E = \overline{V}$. Let $\mu = (\lambda|E)/\lambda(E)$. Let, in Lemma 3, $J = K(E) \cap I$. This is G_δ, hereditary in $K(E)$, and contains all finite subsets of $K(E)$. So, by Lemma 3, since $\hat{\mu}(n) \to 0$, there is, for any given ϵ, N, $\nu \in P(E)$ with $\text{supp}(\nu) \in I \cap K(E)$ and $R(\nu) \leq \frac{(1+\epsilon)}{N}$.

Thus for any open non-empty $V \subseteq \mathbb{T}$ if we choose $x \in V$ we can find for each $n \in \mathbb{N}$ a closed set $K_n \in I$ such that $\text{dist}(x, E_n) < 1/n$ and a probability measure $\nu_n \in P(K_n)$ with $R(\nu_n) \leq \frac{1}{n}$. Let $K = \{x\} \cup \bigcup_n K_n$. Then $K \in I$ and clearly $\inf\{R(\mu) : \mu \in P(K)\} = 0$, so $K \in I \setminus U_0'$.

So it only remains to give the

Proof of Lemma 3. The proof uses methods of Körner and Kaufman.

First, let $J = \bigcap_n G_n$, with G_n decreasing and open in $K(E)$. Let

$$G_n^* = \{K \in K(E) : \forall L \subseteq K(L \in G_n)\} \subseteq G_n.$$

Clearly G_n^* is hereditary and $J = \bigcap_n G_n^*$. It is also easy to see that G_n^* is open too. (We prove that $K(E) \setminus G_n^*$ is closed. Let $K_p \in K(E) \setminus G_n^*$ and $K_p \to K$. Then there exists $L_p \subseteq K_p$ with $L_p \notin G_n$. By compactness, there is a converging subsequence $L_{p_i} \to L$. As $L_{p_i} \to L, K_{p_i} \to K$ and $L_{p_i} \subseteq K_{p_i}$, we have $L \subseteq K$. But $L_{p_i} \notin G_n$, so as G_n is open, $L \notin G_n$, i.e., $K \notin G_n^*$.) So, to avoid complicated notation, we assume that each G_n is open hereditary to start with.

Now notice that if G is open hereditary in $K(E)$ and $K \in G$, then there is open V in E with $K \subseteq V$, so that $K(V) \subseteq G$. (Otherwise for any such V, there is $L_V \in K(V)$ with $L_V \notin G$. Letting V_n be such that $\overline{V_{n+1}} \subseteq V_n$ and $K = \bigcap_n V_n$ and $L_n = L_{V_n}$, we can find a convergent subsequence $L_{n_i} \to L$. Then for any $n, L \subseteq \overline{V_n}$, so $L \subseteq K$ and $L \notin G$, as G is open, a contradiction.)

Before we proceed to the construction of ν we will need a nice observation of Körner.

Lemma 4. *Let* $\Delta = [a, b + \ell]$ *be a closed interval and let* ρ, σ *be probability measures with* $\rho(\Delta) = \sigma(\Delta)$. *Then*

$$\left| \int_\Delta e^{-inx} d\rho - \int_\Delta e^{-inx} d\sigma \right| \leq 2\rho(\Delta) \sup_{x \in \Delta} \left| e^{-inx} - e^{-ina} \right|.$$

Proof. Notice that if $\rho_\Delta = \rho|\Delta, \sigma_\Delta = \sigma|\Delta$, then

$$\int d|\rho_\Delta - \sigma_\Delta| = ||\rho_\Delta - \sigma_\Delta||_M \leq ||\rho_\Delta||_M + ||\sigma_\Delta||_M = \rho_\Delta(\mathbb{T}) + \sigma_\Delta(\mathbb{T}) = 2\rho(\Delta).$$

Now, since $\int d(\rho_\Delta - \sigma_\Delta) = \rho_\Delta(\mathbb{T}) - \sigma_\Delta(\mathbb{T}) = \rho(\Delta) - \sigma(\Delta) = 0$, we have

$$\begin{aligned}
&\left| \int_\Delta e^{-inx} d\rho - \int_\Delta e^{-inx} d\sigma \right| \\
=&\left| \int e^{-inx} d\rho_\Delta - \int e^{-inx} d\sigma_\Delta \right| \\
=&\left| \int (e^{-inx} - e^{ina}) d(\rho_\Delta - \sigma_\Delta) \right| \\
\leq& \sup_{x\in\Delta} |e^{inx} - e^{-ina}| \cdot \int d|\rho_\Delta - \sigma_\Delta| \\
\leq& 2\rho(\Delta) \cdot \sup_{x\in\Delta} |e^{-inx} - e^{ina}|.
\end{aligned}$$

$\dashv$

We are now ready to start the construction: We will define probability measures $\mu_k \in R$ and numbers $p_k \in \mathbb{N}$ such that

(i) $0 < p_0 = p_1 < \cdots < p_{N-1} < p_N < p_{N+1} < \cdots$;

(ii) $\mu_0 = \mu_1 = \cdots = \mu_{N-1} = \mu$;

(iii) $(|j| \leq p_{k+N-1}$ or $|j| \geq p_{k+N}) \Rightarrow |\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)| \leq \frac{1}{2}\epsilon \cdot 2^{-k-1}$;

(iv) $p_k \leq |j| \Rightarrow |\hat{\mu}_k(j)| < \epsilon/2$;

(v) $\text{supp}(\mu_{k+N}) \subseteq \text{supp}(\mu_k)$;

(vi) $\text{supp}(\mu_{n+\ell\cdot N}) \in G_\ell, \ell = 1, 2, \cdots; n = 0, \cdots, N-1$;

(vii) $\text{supp}(\mu_k)$ is a finite union of disjoint closed intervals contained in E, and on each one of these intervals μ_k is a multiple of Lebesgue measure on that interval.

Assume this can be done. For $n = 0, 1, \cdots, N-1$ let

$$\mu^n = \lim_\ell{}^{w^*} \mu_{n+\ell\cdot N} \in P(E).$$

(To see that this limit exists, it is enough to show that $\lim_\ell \hat{\mu}_{n+\ell\cdot N}(j)$ exists for each $j \in \mathbb{Z}$. But, by (iii), given j, if ℓ is so large that $|j| \leq p_{n+\ell\cdot N-1}$, then we have $|\hat{\mu}_{n+\ell\cdot N}(j) - \hat{\mu}_{n+(\ell-1)N}(j)| \leq \frac{1}{2}\epsilon \cdot 2^{-n-(\ell-1)N-1}$, so that $\{\hat{\mu}_{n+\ell\cdot N}(j)\}_\ell$ is a Cauchy sequence.)

By (v), $\text{supp}(\mu^n) \subseteq \text{supp}(\mu_{n+\ell N})$ for any ℓ, so $\text{supp}(\mu^n) \in \bigcap_\ell G_\ell = J$. Let $\nu = \frac{1}{N}(\mu^0 + \cdots \mu^{N-1})$. Then $\text{supp}(\nu) \subseteq \bigcup_{n=0}^{N-1} \text{supp}(\mu^n)$, so, as J is hereditary, $\text{supp}(\nu) =$

$E_0 \cup \cdots \cup E_{N-1}$ with $E_n \in J$. Finally, fix $j \in \mathbb{Z}$, in order to show that $|\hat{\mu}(j) - \hat{\nu}(j)| \leq \frac{(1+\epsilon)}{N}$. We have

$$
\begin{aligned}
&|\hat{\mu}(j) - \hat{\nu}(j)| \\
=&\left| \frac{1}{N} \left(\sum_{n=0}^{N-1} (\hat{\mu}_n(j) - \lim_{\ell} \hat{\mu}_{n+\ell N}(j)) \right) \right| \\
=&\lim_{\ell} \left| \frac{1}{N} \sum_{n=0}^{N-1} (\hat{\mu}_n(j) - \hat{\mu}_{n+\ell N}(j)) \right| \\
\leq&\frac{1}{N} \sum_{k=0}^{\infty} |\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)|.
\end{aligned}
$$

Now if $|j| \leq p_{k+N-1}$ or $|j| \geq p_{k+N}$, we have

$$|\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)| \leq \frac{\epsilon}{2} \cdot 2^{-k-1},$$

and if $p_{k+N-1} < |j| < p_{k+N}$, then $|\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)| \leq 1 + \epsilon/2$, since $|\hat{\mu}_k(j)| < \epsilon/2$ for $|j| \geq p_k$ and $n_{k+N-1} > n_k$. So

$$
\begin{aligned}
|\hat{\mu}(j) - \hat{\nu}(j)| \leq& \frac{1}{N}(1 + \epsilon/2 + \sum_{k=0}^{\infty} \epsilon/2 \cdot 2^{-k-1}) \\
=& \frac{(1+\epsilon)}{N}.
\end{aligned}
$$

To construct μ_n, p_n satisfying (i) – (vii) we start with $\mu_0 = \mu_1 = \cdots \mu_{N-1} = \mu$ and $0 < p_0 = p_1 = \cdots = p_{N-1}$, so that $|j| \geq p_0 \Rightarrow |\hat{\mu}(j)| < \epsilon/2$ (which can be found as $\hat{\mu}(j) \to 0$.)

Now we assume the construction has been done up to $k + N - 1$ ($k = 0, 1, 2, \cdots$), and we will construct μ_{k+N}, p_{k+N}. Let $\text{supp}(\mu_k) = \Delta_1 \cup \cdots \cup \Delta_{r_k}, \Delta_m$ pairwise disjoint closed intervals contained in E. Let also $k = n + \ell N$ ($0 \leq n \leq N-1, \ell \geq 0$), so that $k + N = n + (\ell+1)N$.

Using Lemma 4, split each Δ_m into finitely many small enough closed subintervals, $\Delta_{m,q}$, with only endpoints in common, so that the oscillation of e^{-ijx} for $|j| \leq n_{k+N-1}$ in each one of them is $\leq \frac{1}{4}\epsilon \cdot 2^{-k-1}$. Then by Lemma 4, if ρ, σ are continuous (i.e., $\rho(\{x\}) = \sigma(\{x\}) = 0$) probability Borel measures supported by $\bigcup_m \Delta_m = \bigcup_{m,q} \Delta_{m,q}$ and $\rho(\Delta_{m,q}) = \sigma(\Delta_{m,q})$ for every m, q, then we have

$$|j| \leq p_{k+N-1} \Rightarrow |\hat{\rho}(j) - \hat{\sigma}(j)| \leq \frac{1}{2}\epsilon \cdot 2^{-k-1}.$$

Choose one point $x_{m,q}$ in the interior of each $\Delta_{m,q}$ and denote the resulting finite set by $K = \{x_{m,q}\}$. Then $K \in G_{\ell+1}$, so, as $G_{\ell+1}$ is open hereditary, we can find a small closed interval $\Gamma_{m,q}$ contained in the interior of $\Delta_{m,q}$ with $x_{m,q} \in \Gamma_{m,q}$, so that all closed subsets of $\bigcup_{m,q} \Gamma_{m,q}$ are contained in $G_{\ell+1}$. We can of course assume that the $\Gamma_{m,q}$ are pairwise disjoint.

Define then the probability Borel measure μ_{k+N} as follows: μ_{k+N} has support $\bigcup_{m,q} \Gamma_{m,q}$ and

$$\mu_{k+N}|\Gamma_{m,q} = \mu_k|\Gamma_{m,q} \cdot \frac{\mu_k(\Delta_{m,q})}{\mu_k(\Gamma_{m,q})}.$$

It is clear that (v)–(vii) are true. Now

$$\mu_{k+N}(\Delta_{m,q}) = \mu_k(\Gamma_{m,q}) \cdot \frac{\mu_k(\Delta_{m,q})}{\mu_k(\Gamma_{m,q})}$$

$$= \mu_k(\Delta_{m,q}),$$

so (iii) holds for $|j| \leq p_{k+N-1}$. Finally, choose $p_{k+N} > p_{k+N-1}$ large enough so that $|j| \geq p_{k+N} \Rightarrow |\hat{\mu}_{k+N}(j)| < \epsilon/2($ which can be done since $\hat{\mu}_{k+N}(j) \to 0$, as $|j| \to \infty)$, and also $|j| \geq p_{k+N} \Rightarrow |\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)| \leq \frac{1}{2}\epsilon \cdot 2^{-k-1}$ (which again can be done since $\hat{\mu}_k(j) \to 0$ as $|j| \to \infty$).

This completes the construction, the proof and these lectures. $\dashv$

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