Volumes of Tubes in Hyperbolic 3-Manifolds

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1 Introduction

In this paper we establish the first step of a program that we hope will determine the lowest volume closed hyperbolic 3-manifolds.

The interest in volumes arises for several reasons. By Mostow Rigidity, the volume of a complete finite-volume hyperbolic 3-manifold is a topological invariant. Thurston [T1], building on work of Gromov and Jørgensen, showed that the order type of the set of volumes of complete finite volume hyperbolic 3-manifolds is ω^ω and that at most finitely many manifolds can have the same volume.

In 1987 Adams [A] showed that the Gieseking manifold is the smallest volume complete noncompact hyperbolic 3-manifold. More recently, Cao and Meyerhoff [CM] showed that the figure-8 knot complement and its sibling are the two smallest complete orientable noncompact hyperbolic 3-manifolds. The figure-8 knot complement has volume $2.02\ldots$ and double covers the Gieseking manifold.

For closed 3-manifolds the situation is far less well understood. The Weeks manifold $\mathcal{W}$, which has volume $0.9427\ldots$, is widely believed to be the closed hyperbolic 3-manifold of least volume. By Culler, Hersonsky, and Shalen [CHS] if N is a closed orientable hyperbolic 3-manifold with first Betti number greater than 2, then $\mathrm{Vol}(N) > \mathrm{Vol}(\mathcal{W})$. By Przeworski [P] if S is a smallest volume closed orientable hyperbolic 3-manifold, then $\mathrm{Vol}(S) > 0.27$. The first lower bound for this number was 0.00064 [M1], but thanks to the combination of Gehring and Martin [GM] and Gabai, Meyerhoff, and Thurston [GMT] this number was raised to $0.166\ldots$. Recently Przeworski [P] sharpened this result to obtain the .27 bound.

The main result of this paper shows that there is a lower bound on the lengths of geodesics in a smallest volume closed orientable hyperbolic 3-manifold.

Theorem 1.1 *If W is a maximal tube of radius r about a geodesic γ in the complete orientable hyperbolic 3-manifold M and either* $\mathrm{length}(\gamma) \leq 0.069$ *or* $r \geq 1.483$*, then* $\mathrm{volume}(W) \geq 0.95$.

Corollary 1.2 *A shortest geodesic in a least-volume closed orientable hyperbolic 3-manifold has length greater than* 0.069.

The authors and Nathaniel Thurston can now begin a computer investigation of low volume closed hyperbolic 3-manifolds in a spirit similar to [GMT], for Corollary 1.2 guarantees that the needed parameter space is compact.

The remainder of this section outlines the proof of Theorem 1.1 and provides a very elementary explanation as to why such a result should be true. It also introduces ideas and facts which may be of independent interest. The detailed proof of Theorem 1.1 is given in §2 - §5.

1.1 Definitions

Let γ be a simple closed geodesic in the complete orientable hyperbolic 3-manifold $M = \mathbb{H}^3/G$ where G is identified with $\pi_1(M)$. Let γ_i denote the lifts of γ to the universal covering $\mathbb{H}^3$ with γ_0 denoting a fixed lift. We say that two lifts γ_i, γ_j are *conjugate* if there exists a $w \in \pi_1(M)$ such that $w(\gamma_i) = \gamma_0$ and $w(\gamma_0) = \gamma_j$. Let $H \subset \pi_1(M)$ be the subgroup generated by γ. We can assume that it is the subgroup of $\pi_1(M)$ which stabilizes γ_0. Partition $\{\gamma_i\} - \gamma_0$ into equivalence classes called *orthoclasses* by saying that γ_i is equivalent to γ_j if either γ_i is conjugate to γ_j or $\gamma_i = h(\gamma_j)$ where $h \in H$.

Lemma 1.3 *Each orthoclass contains exactly two H-orbits.*

Proof: Since H is the stabilizer of γ_0, an orthoclass contains at most 2 H-orbits. Conversely if γ_i and γ_j are conjugate via w, then γ_i and γ_j lie in distinct H-orbits (a fact first proven for horoballs by Adams [A]). To see this let α denote the oriented geodesic arc from γ_i to γ_0. Then $w(\alpha)$ is the oriented geodesic arc from γ_0 to γ_j. If $\gamma_j = h(\gamma_i)$ for $h \in H$, then $w^{-1}h(\alpha) = -\alpha$, where $-\alpha$ denotes α with the opposite orientation. Thus $w^{-1}h$ is a nontrivial covering transformation that has a fixed point, which is a contradiction. $\square$

Associated to an orthoclass is a positive real number which is the real distance from any element in that class to γ_0. Let $\mathcal{O}(1), \mathcal{O}(2), \ldots$ denote the orthoclasses ordered so that if $O(i)$ denotes the corresponding distance, then $O(1) \leq O(2) \leq \cdots$. The solid tube V_0 of radius $r = O(1)/2$ projects to a tube W in M with the property that $\overset{\circ}{W}$ is embedded and W is only immersed. Thus W is called a *maximal tube* and $r = O(1)/2$ is called the *tuberadius* of γ.

For each i, let V_i denote the tube of radius $O(1)/2$ about γ_i. The orthoclass equivalence relation of $\{\gamma_i\} - \gamma_0$ induces an equivalence relation on $\{V_i\} - V_0$. Call such an equivalence class an *orthotube class* and let $\mathcal{OT}(k)$ denote the class corresponding to $\mathcal{O}(k)$. Define $OT(k) = O(k) - O(1)$ which equals the distance from V_0 to any element in $\mathcal{OT}(k)$. The utility of studying the *orthotube spectrum* (which is the sequence $OT(1)$, $OT(2), \ldots$), rather than the *ortholength spectrum* is that the concept of orthotube spectrum generalizes in the obvious way to the notion of *orthohoroball spectrum* for noncompact complete hyperbolic 3-manifolds. It is useful to think of manifolds with very thick tubes as being geometrically close to cusped manifolds, and hence to treat them somewhat like cusped manifolds.

The following useful formulas are well known and probably go back to Lobachevsky, for example see Fenchel's book [F].

Lemma 1.4 *Let W be a tube of radius r about a geodesic of length l. Then*

- area$(\partial W) = 2\pi l \sinh(r)\cosh(r)$
- volume$(W) = \pi l \sinh^2(r)$ □

The volume of W is a lower bound for the volume of M. To estimate the volume of W use the fact that if tuberadius(γ) is large (or length(γ) small and hence tuberadius(γ) large [M1]), then volume(W) is approximately equal to area$(\partial W)/2$. Intuitively a very thick tube is geometrically very much like a horotorus and the volume of a horotorus T is exactly area$(\partial T)/2$. The following exact formulas reinforce this fact, although they will not be used in this paper:

Lemma 1.5 *If W is a tube of radius r about a closed geodesic γ in a hyperbolic 3-manifold such that $\mathring{W}$ is embedded, then*

1) volume$(W) = \frac{1}{2}$area$(\partial W) - \frac{\pi}{2}$length$(\gamma) + \frac{\pi}{4}$length$(\gamma)e^{-2r}$
2) [GM] volume$(W) = \frac{1}{2}\tanh(r)$area$(\partial W)$

Proof of 1) Use the fact that area$(\partial W) = 2\pi l \sinh(r)\cosh(r) = 2\pi l(e^r - e^{-r})(e^r + e^{-r})/4 = \pi l(e^{2r} - e^{-2r})/2$ and that Volume$(W) = \pi l \sinh^2(r) = \pi l e^{2r}/4 - \pi l/2 + \pi l e^{-2r}/4$. □

The idea of the proof of Theorem 1.1 is as follows. As in [GM] or [P] call the orthogonal projection V_i to ∂V_0 the *shadow* of V_i, denoted $S(V_i)$. By [P] the shadows of the various elements of a given orthotube class have pairwise disjoint interiors, provided that $r \geq \log 3/2$. Thus by Lemma 1.3, twice the area of $S(V_i)$ is a lower bound for area(∂W). The proof of Theorem 1.1 falls into two main steps.

Step 1. If $OT(2) \geq .3$ and length$(\gamma) \leq 0.069$, then volume$(M) > 0.95$.

Why you should believe this. In the limiting case where W corresponds to a horoball, the shadow is a round disk of area $\pi/4$. By doubling the area, using Lemma 1.3, and disk packing one obtains the asymptotic area estimate of $\sqrt{3}/\pi \approx 2(0.86\ldots)$ and hence volume(W) is asymptotically bounded below by $0.86\ldots$.

When $OT(2) > 0$ improve this number as follows. Lemma 3.2 shows that if $A, B \in \mathcal{OT}(1)$ and $A \neq B$, then $A \cap B = \emptyset$ which implies that $d(A,B) \geq OT(2)$. Therefore if $OT(2) \geq 0.3$, and V_i', V_j' are tubes of radius $O(1)/2 + 0.15$ about distinct elements in $\mathcal{O}(1)$, then their orthogonal projections to ∂V have pairwise disjoint interiors. In the limit when W corresponds to a horoball, these projections are round discs of area $> \pi(0.337\ldots)$. The usual disk packing argument now implies that volume$(W) > 1.164\ldots$.

Step 2. If $OT(2) \leq .3$, and length$(\gamma) \leq .069$, then volume$(M) > .95$.

Why you should believe this. Here the shadows of the $\mathcal{OT}(1) \cup \mathcal{OT}(2)$ tubes are nearly disjoint and of large area. Indeed asymptotically when W is a horotorus, the shadow of an $\mathcal{OT}(1)$ horoball is a round disk of area $\pi/4$ and the shadow of an $\mathcal{OT}(2)$ horoball contains a round disk of area $\pi(.12)$ which is disjoint from the $\mathcal{OT}(1)$ shadows. Thus asymptotically, volume$(W) = $ area$(\partial W)/2 > .5(2(\pi/4 + .12\pi)) > 1.16$.

The proof of Theorem 1.1 is organized as follows. In §2 a lower bound of tuberadius(γ) is given in terms of length(γ). This is sharpened in §3 where a lower bound of tuberadius(γ) is given in terms of length(γ) and $OT(2)$, for $OT(2)$ sufficiently large. When $OT(2)$ is not sufficiently large, then in §4 another lower bound of tuberadius(γ) is given in terms of length(γ) and $OT(2)$. By Lemma 1.4, the main results of §2 - §4 give lower bounds of volume(W) as a function of length(γ). Using these estimates the proof of Theorem 1.1 is completed in §5.

2 The first order tube radius formula

This section is devoted to proving the following:

Proposition 2.1 *If γ is a geodesic of length l and tube radius r in a complete orientable hyperbolic 3-manifold, then*

$$l \geq \frac{\sqrt{3}\cosh(2r)}{2\pi\sinh(2r)}\left(\cosh^{-1}\left(\frac{\sinh^2(2r)+\cosh(2r)}{\cosh^2(2r)}\right)\right)^2 \tag{2.1}$$

Furthermore if $l \leq 0.10438$, then the right side of (2.1) is invertible and we obtain an implicit lower bound on r as a function of l.

For any tube $V_j \neq V_0$, define the *center* v_j of V_j to be the point on γ_j which is closest to γ_0. Thus the center of a tube in $\mathcal{OT}(n)$ must lie at a distance of $O(n)$ from γ_0.

Let $C \subset \mathbb{H}^3$ be the boundary of the cylinder of radius $2r$ about γ_0. Note that C is a Euclidean surface, isometric to the boundary of a right circular cylinder in $\mathbb{E}^3$ of radius $\sinh(2r)$, and C contains the centers of all tubes in $\mathcal{OT}(1)$. In what follows both the hyperbolic distance between points on C and the distance along the surface of C will appear. So to avoid ambiguity use the notation $d_H(p,q)$ to refer to the hyperbolic metric and $d_E(p,q)$ to refer to the restriction of the metric to C.

2.1 Euclidean and Cylindrical Coordinates

Define a pair of coordinate systems for C as follows. Choose an origin $(0,0)$, and for any d and ϕ let (d,ϕ) be the image of $(0,0)$ under the loxodromic motion with axis γ_0 and complex length $d+i\phi$. (This assumes a choice of orientation has already been made for γ_0.) Any point of C has coordinates (d,ϕ) where d is unique and ϕ is unique modulo 2π. Define a second pair of coordinates (x,y) on C by the following relations: $x = d\cosh(2r)$ and $y = \phi\sinh(2r)$. It is a simple exercise in hyperbolic trigonometry to show that x and y are natural Euclidean coordinates for C, in the sense that if $p=(x_1,y_1)$ and $q=(x_2,y_2)$, and if $|y_1-y_2| \leq \pi\sinh(2r)$, then $d_E(p,q)^2 = (x_1-x_2)^2+(y_1-y_2)^2$. Hence, refer to x and y as *Euclidean* coordinates for C, and refer to d and ϕ as *cylindrical* coordinates. Unless stated otherwise, from this point on assume that $-\pi < \phi \leq \pi$ and equivalently $-\pi\sinh(2r) < y \leq \pi\sinh(2r)$.

While Euclidean coordinates capture the Euclidean geometry of C, cylindrical coordinates capture the hyperbolic geometry of C as follows:

Lemma 2.2 *The hyperbolic distance between two points p_0, $p_1 \in C$ is given by the formula*

$$\cosh(d_H(p_0,p_1)) = \cosh(d_1-d_0)\cosh^2(2r) - \cos(\phi_1-\phi_0)\sinh^2(2r) \tag{2.2}$$

where (d_0,ϕ_0) and (d_1,ϕ_1) are the cylindrical coordinates of p_0 and p_1 respectively.

Proof: Assume that $d_0=\phi_0=0$; the proof in the general case is similar. Consider the Klein hyperboloid model of hyperbolic three-space. In this model, $\mathbb{H}^3$ is the hypersurface

$$\{(x,y,z,t) \in \mathbb{R}^4 | -x^2-y^2-z^2+t^2 = 1,\ t>0\}$$

and geodesics are just the intersection of $\mathbb{H}^3$ with planes through the origin in $\mathbb{R}^4$. The orientation-preserving isometries of this model are given by the connected component of the identity in the Lie group

$$O(3,1) = \{A \in \mathrm{GL}(4,\mathbb{R}) | A^tQA = Q\}$$

where Q is the diagonal matrix with diagonal entries $(1,1,1,-1)$. Elements of this group act on points in $\mathbb{H}^3 \subset \mathbb{R}^4$ by matrix multiplication on the left (view points as column vectors). The metric d_H is given by the formula

$$\cosh^{-1}(-x_1x_2-y_1y_2-z_1z_2+t_1t_2) \tag{2.3}$$

(For more details about the hyperboloid model, see Thurston's book [T2].)

Using this model, without loss of generality assume that γ_0 is the intersection of $\mathbb{H}^3$ with the plane $\{y=z=0\}$. Then a loxodromic motion along γ_0 with complex length $d+i\phi$ is given by the matrix

$$\begin{pmatrix} \cosh(d) & 0 & 0 & \sinh(d) \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ \sinh(d) & 0 & 0 & \cosh(d) \end{pmatrix}$$

Assume further that the origin of C is at the point $(0,\ 0,\ \sinh(2r),\ \cosh(2r))$. Then by definition the point on C with cylindrical coordinates (d_1, ϕ_1) must be the point

$$(\sinh(d_1)\cosh(2r), -\sin(\phi_1)\sinh(2r), \cos(\phi_1)\sinh(2r), \cosh(d_1)\cosh(2r))$$

Hence by the distance formula for the hyperboloid model,

$$\cosh(d_H(p_0, p_1)) = \cosh(d_1)\cosh^2(2r) - \cos(\phi_1)\sinh^2(2r)$$

as desired. $\square$

Lemma 2.2 admits an easy generalization which is also useful, and the proof of which will be left to the reader:

Lemma 2.3 *If q is a point outside of C such that $d(q, C) = \rho$ (that is, if $d(q, \gamma_0) = 2r + \rho$), and such that the projection of q to C has cylindrical coordinates $(0, 0)$, and if $p \in C$, then the distance from q to p is given by*

$$\cosh(d_H(p, q)) = \cosh(d)\cosh(2r)\cosh(2r + \rho) - \cos(\phi)\sinh(2r)\sinh(2r + \rho) \tag{2.4}$$

where (d, ϕ) are cylindrical coordinates for p. $\square$

2.2 The Fundamental Domain of H on C

Choose some $U \in \mathcal{OT}(1)$, and let the center u of U be the origin in C. The images of u under H are the vertices of a tiling of C by quadrilaterals. The area A of a fundamental region for the action of H on C is the area of one of these quadrilaterals. However, A is also the area of the boundary of the manifold $\bar{C}/H$, where $\bar{C}$ is the solid cylinder bounded by C. Hence by Lemma 1.4,

$$A = 2\pi l \cosh(2r)\sinh(2r)$$

A lower bound for A in terms of r alone will give a lower bound for l in terms of r. Suppose δ_0 is any lower bound for the distance $d_E(p, q)$ between the centers of any two tubes in $\mathcal{OT}(1)$.

Then each such center is contained in circle of radius $\delta_0/2$ containing no other such centers, and any two such circles have disjoint interiors.

Note however that any fundamental region for the action of H contains the centers of at least two tubes in $\mathcal{OT}(1)$, one from each H-orbit. Thus,

$$A \geq 2\pi \left(\frac{\delta_0}{2}\right)^2$$

As in [A] and [M2], improve the above estimate by a constant factor as follows: a packing of C by circles of radius $\delta_0/2$ lifts to a circle packing of the Euclidean plane, which can be no denser than the hexagonal packing. The hexagonal packing of the plane has density $\pi/(2\sqrt{3})$, hence

$$\begin{aligned} A &\geq \frac{\pi {\delta_0}^2}{2} \frac{2\sqrt{3}}{\pi} \\ &= {\delta_0}^2 \sqrt{3} \end{aligned}$$

And hence

$$l \geq \frac{{\delta_0}^2\sqrt{3}}{2\pi \cosh(2r)\sinh(2r)} \tag{2.5}$$

It remains to find an estimate for δ_0 in terms of r.

2.3 The Distance Between Centers

Let p be the center of any tube in $\mathcal{OT}(1)$ other than the initial tube U. Since u and p lie on the axes of two separate tubes of radius r, $d_H(u,p) \geq 2r$. Hence $d_E(u,p) \geq 2r$. However for large r this estimate will be poor; a closer examination of the situation is required.

For any $t > 0$ and any $\rho \geq 0$, let $B(t,\rho)$ be the region consisting of all points $c \in C$ such that $d_H(c,q) \leq t$, where q is the unique point in $\mathbb{H}^3$ at a distance of $2r+\rho$ from γ_0 and whose projection down to C equals u. Note that the boundary of $B(t,\rho)$ is the intersection of C with the sphere of radius t centered at q; it is easy to see that for $\rho < t < 4r+\rho$, the boundary of $B(t,\rho)$ will be a single closed curve. The following lemma describes the largest disk centered at $u \in C$ and contained in $B(t,\rho)$.

Lemma 2.4 *If $0 \leq \rho < t < 4r + \rho$, then the region $B(t,\rho)$ contains a disk centered at u with radius*

$$R_{t,\rho} = \sqrt{\sinh(2r)\cosh(2r)\coth(2r+\rho)}\cosh^{-1}\left(\frac{\sinh(2r)\sinh(2r+\rho)+\cosh(t)}{\cosh(2r)\cosh(2r+\rho)}\right) \tag{2.6}$$

Note that this lemma immediately implies that $R_{2r,0}$ is a choice for the lower bound for δ_0.

Proof: By lemma 2.3, the boundary of $B(t,\rho)$ can be described by the equation

$$\cosh(d)\cosh(2r)\cosh(2r+\rho) - \cos(\phi)\sinh(2r)\sinh(2r+\rho) = \cosh(t)$$

where d and ϕ are cylindrical coordinates on C. Since $|\cos(\phi)| \leq 1$,

$$|d| \leq \cosh^{-1}\left(\frac{\sinh(2r)\sinh(2r+\rho)+\cosh(t)}{\cosh(2r)\cosh(2r+\rho)}\right) \tag{2.7}$$

Let a be the quantity on the right-hand side of equation (2.7).

The next step is to find a lower bound on ϕ in terms of d. The following derivation is based on a similar derivation in [GM]. By elementary calculus, for all $|d| \leq a$

$$\begin{aligned} \cosh(a) - \cosh(d) &\geq \frac{a^2-d^2}{2} \\ \Rightarrow \frac{\sinh(2r)\sinh(2r+\rho)+\cosh(t)}{\cosh(2r)\cosh(2r+\rho)} - \cosh(d) &\geq \frac{a^2-d^2}{2} \end{aligned}$$

Rearranging the last inequality, for any point (d,ϕ) on the boundary of $B(t,\rho)$ we get

$$\begin{aligned} 1 - \frac{\cosh(2r)\cosh(2r+\rho)}{\sinh(2r)\sinh(2r+\rho)}\left(\frac{a^2-d^2}{2}\right) &\geq \frac{\cosh(d)\cosh(2r)\cosh(2r+\rho)-\cosh(t)}{\sinh(2r)\sinh(2r+\rho)} \\ &= \cos(\phi) \end{aligned}$$

By another application of elementary calculus, the left-hand side of the above inequality is less than

$$\cos\left(\sqrt{\coth(2r)\coth(2r+\rho)(a^2-d^2)}\right)$$

Hence

$$\cos\left(\sqrt{\coth(2r)\coth(2r+\rho)(a^2-d^2)}\right) \geq \cos(\phi) \tag{2.8}$$

Recall that $-\pi < \phi \leq \pi$. Since $\phi = 0$ when $|d| = a$, and since the boundary of $B(t,\rho)$ is a single continuous curve, for all points (d,ϕ) on the boundary of $B(t,\rho)$

$$|\phi| \geq \sqrt{\coth(2r)\coth(2r+\rho)(a^2-d^2)}$$

Converting into Euclidean coordinates, $|x| \leq a\cosh(2r)$ and

$$|y| = |\phi\sinh(2r)| \geq \sqrt{\tanh(2r)\coth(2r+\rho)}\sqrt{(a\cosh(2r))^2-x^2}$$

for any (x, y) on the boundary of $B(t, \rho)$. Thus the region $B(t, \rho)$ contains an ellipse with axes parallel to the coordinate axes. (The ellipse will be a circle if $\rho = 0$.) Since tanh is an increasing function, $\tanh(2r) \leq \tanh(2r + \rho)$ and hence

$$\tanh(2r) \coth(2r + \rho) \leq 1$$

Hence the minor axis of this ellipse is in the y-direction. Thus the region $B(t, \rho)$ contains a circle centered at u with radius

$$a\sqrt{\sinh(2r) \cosh(2r) \coth(2r + \rho)} \tag{2.9}$$

This proves the lemma. □

In particular, $B(2r, 0)$ contains a circle of radius $R_{2r,0}$ centered at u, and hence δ_0 may be chosen as follows:

$$\delta_0 = \cosh(2r) \cosh^{-1}\left(\frac{\sinh^2(2r) + \cosh(2r)}{\cosh^2(2r)}\right) \tag{2.10}$$

Substituting this into equation (2.5) results in equation (2.1), which proves the first part of Proposition 2.1.

2.4 Bounding r in terms of l

A quick computer examination of the function on the right-hand side of equation (2.1) shows that it has the asymptotic properties one would expect. Namely, as r goes to infinity the lower bound on l goes to 0 and the corresponding lower bound on the volume of W (which is just $\pi l \sinh^2(r)$ by Lemma 1.3) approaches $\sqrt{3}/2$. However the function's behavior near 0 is more problematic: the function appears to go to 0 as $r \to 0^+$ and seems to have a local maximum near $r = 0.7$. Thus the equation (2.1) alone does not give a lower bound for r in terms of l.

To get around this problem, use an earlier theorem by Meyerhoff and Zagier [M1]:

Theorem 2.5 (Meyerhoff and Zagier,1987) *Let γ be a geodesic in a complete hyperbolic 3-manifold. If the real length l of γ is less than*

$$\frac{\sqrt{3}}{4\pi}\left(\log(\sqrt{2}+1)\right)^2 \approx 0.107$$

then there exists an embedded solid tube around γ whose radius r satisfies

$$\sinh^2(r) = \frac{1}{2}\left(\frac{\sqrt{1-2k}}{k} - 1\right)$$

where

$$k = \cosh\left(\sqrt{\frac{4\pi l}{\sqrt{3}}}\right) - 1$$

The above theorem gives a lower bound for r in terms of l, namely

$$r \geq \sinh^{-1}\left(\sqrt{\frac{\sqrt{3 - 2\cosh\left(\sqrt{\frac{4\pi l}{\sqrt{3}}}\right)}}{2\cosh\left(\sqrt{\frac{4\pi l}{\sqrt{3}}}\right) - 2} - \frac{1}{2}}\right)$$

and given that $l > 0$, the above expression can be written as

$$l \geq \frac{\sqrt{3}}{4\pi}\left(\cosh^{-1}\left(1 + \frac{-1 + \sqrt{2 + 4\sinh^2(r) + 4\sinh^4(r)}}{\left(1 + 2\sinh^2(r)\right)^2}\right)\right)^2 \tag{2.11}$$

Comparing the expression on the right hand side of (2.11) with that on the right hand side of (2.1), one sees that Meyerhoff's estimate of l is greater than the one from (2.1) when $r < 0.21244$, at which point both estimates say that $l \geq 0.10438$. Furthermore, an examination of equation (2.1) by computer indicates that the function on the right hand side of (2.1) is increasing for r less than 0.52396, and decreasing for r greater than 0.52397, and the function equals 0.10438 when r equals $0.212439\ldots$ or $1.02617\ldots$.

Hence for $r > 1.02618$ and $l < .10438$ the function on the right hand side of (2.1) is invertible. Hence equation (2.1) can be expressed as a lower bound on r in terms of l which is valid when $l \leq 0.10438$. This completes the proof of Proposition 2.1.

The asymptotic value of $l\pi\sinh^2(r)$ given by this estimate when $l \to 0^+$ (or equivalently when $r \to \infty$) is $\sqrt{3}/2$. Note that the corresponding asymptotic value obtained from Meyerhoff's estimate alone is $\sqrt{3}/4$, so the combined result is an improvement by a factor of two. Unfortunately, $\sqrt{3}/2$ is approximately 0.866 while the lowest-volume hyperbolic three-manifold found so far has a volume of approximately 0.943. So it is necessary to find ways to improve the result of Proposition 2.1 in particular cases.

For example, if $OT(2) = 0$ then Lemma 1.3 together with the proof of Proposition 2.1 yields

Proposition 2.6 *If γ is a geodesic of length l and tuberadius r in a complete orientable hyperbolic 3-manifold, and if $\mathcal{OT}(2) = 0$, then*

$$l \geq \frac{\sqrt{3}\cosh(2r)}{\pi\sinh(2r)}\left(\cosh^{-1}\left(\frac{\sinh^2(2r) + \cosh(2r)}{\cosh^2(2r)}\right)\right)^2 \tag{2.12}$$

Furthermore if $l \leq 0.10438$, then the right side of formula (2.12) is invertible and provides an implicit lower bound on r as a function of l.

Note that the asymptotic volume of W obtained from Proposition 2.6 when $l \to 0^+$ is $\sqrt{3}$. (This proposition is not used in the rest of this paper; instead, Proposition 4.1 is used.)

3 The second order tube radius formula

In this section the result of Proposition 2.1 is improved when $OT(2)$ is sufficiently large.

Proposition 3.1 *If γ is a geodesic of length l and tuberadius r in a complete orientable hyperbolic 3-manifold and if $OT(2) > \rho$, where $0 < \rho < 2r$, then*

$$l \geq \frac{\sqrt{3}\cosh(2r)}{2\pi\sinh(2r)}\left(\cosh^{-1}\left(\frac{\sinh^2(2r)+\cosh(2r+\rho)}{\cosh^2(2r)}\right)\right)^2 \tag{3.1}$$

Furthermore if $\rho = 0.3$ then the right-hand side of formula (3.1) is strictly decreasing and hence invertible, and provides an implicit lower bound on r as a function of l.

Proof: Let A and B be distinct tubes in $\mathcal{OT}(1)$. Since $OT(2) > \rho > 0$, by Lemma 3.2 at the end of this section $d(A,B) \geq \rho$.

Hence the center of each tube in $\mathcal{OT}(1)$ is contained in an open ball of hyperbolic radius $2r+\rho$ which contains the center of no other tube in $\mathcal{OT}(1)$. Define the cylinder C as in the previous section, and consider the ball of hyperbolic radius $2r+\rho$ centered at the origin of C. The intersection of this ball with the cylinder C forms the region $B(2r+\rho,0)$. By Lemma 2.4, the region $B(2r+\rho,0)$ contains a disk of radius

$$\cosh(2r)\cosh^{-1}\left(\frac{\sinh^2(2r)+\cosh(2r+\rho)}{\cosh^2(2r)}\right)$$

Substituting this new estimate into equation (2.5) proves the first part of Proposition 3.1. The proof that the right hand side of (3.1) is decreasing when $\rho = 0.3$ is a matter of computer analysis.

In particular, the estimate of Proposition 3.1 when $\rho = 0.3$ says $\pi l \sinh^2(r) \geq 0.9509$ when $r \geq 1.478$, and that $r \geq 1.478$ when $l \leq 0.07$.

It remains to prove the following:

Lemma 3.2 *If $A, B \in \mathcal{OT}(1)$ and $A \neq B$, then either*

- *$OT(2) = 0$, or*
- *$d(A,B) \geq OT(2)$*

Proof: Suppose that $d(A, B) < OT(2)$ and that $OT(2) > 0$. Let $w \in \pi_1(M)$ be such that $w(A) = V_0$. Then $d(V_0, w(B)) < OT(2)$, which implies $w(B) \in \mathcal{OT}(1)$. In other words $w(B)$ and V_0 are tangent and consequently B and $w^{-1}(V_0) = A$ are tangent. This will imply the existence of an elliptic element of order 3 in $\pi_1(M)$, a contradiction.

There are two cases to consider. Either A and B lie in the same H-orbit, or they lie in conjugate H-orbits. These cases will be handled in turn in the next two sections, which will complete the proof.

3.1 A and B lie in the same H-orbit

Suppose that A and B lie in the same H-orbit, that is $B = \sigma(A)$ where $\sigma \in H$. Since $w(B) \in \mathcal{OT}(1)$, $w(B)$ is a translate of either A or $w(V_0)$ under H by Lemma 1.3.

Suppose that $w(B) = \alpha(A)$, $\alpha \in H$. Then $w\sigma w^{-1}(V_0) = \alpha w^{-1}(V_0)$, and hence $w\sigma w^{-1} = \alpha w^{-1}\beta$ for some $\beta \in H$. At this point, resort to direct computation in the upper half-space model of hyperbolic three-space. Then α, β, σ, and w are all elements of $\mathrm{PSL}(2, \mathbb{C})$. In addition make the following assumptions without loss of generality:

- The line γ_0 is the line from 0 to ∞, and hence all elements of H have this line as their axis.
- The geodesic arc from γ_0 to the axis of A lies on the line from 1 to -1.
- The axis of w is the line from 1 to -1.

Furthermore by multiplying β by $-I$ if necessary assume that $w\sigma w^{-1} = \alpha w^{-1}\beta$ as matrices in $\mathrm{SL}(2, \mathbb{C})$. Then α, β, and σ will be diagonal matrices with diagonal entries (a, a^{-1}), (b, b^{-1}), and (s, s^{-1}) respectively, while

$$w = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

for some x, y satisfying $x^2 - y^2 = 1$. Making the appropriate substitutions, the matrix equation $w\sigma w^{-1} = \alpha w^{-1}\beta$ becomes

$$\begin{pmatrix} sx^2 - s^{-1}y^2 & xy(s^{-1} - s) \\ xy(s - s^{-1}) & s^{-1}x^2 - sy^2 \end{pmatrix} = \begin{pmatrix} abx & -ab^{-1}y \\ -a^{-1}by & a^{-1}b^{-1}x \end{pmatrix}$$

Taking the products of the (1,2) and (2,1) entries on each side,

$$-x^2y^2(s - s^{-1})^2 = y^2$$

Note that if $y = 0$ then $w = \pm I$, a contradiction. So $y \neq 0$, and hence $x^2(s - s^{-1})^2 = -1$. Now consider the group element $\sigma w \sigma w^{-1}$. By direct calculation,

$$\begin{aligned} \operatorname{tr}(\sigma w \sigma w^{-1}) &= (s^2 x^2 - y^2) + (s^{-2} x^2 - y^2) \\ &= (s^2 + s^{-2}) x^2 - 2(x^2 - 1) \\ &= (s - s^{-1})^2 x^2 + 2 \\ &= 1 \end{aligned}$$

Hence $\sigma w \sigma w^{-1}$ is elliptic of order 3, a contradiction.

Now suppose instead that $w(B) = \alpha w(V_0)$ for some $\alpha \in H$. Then $w \sigma w^{-1} = \alpha w \beta$ for some $\beta \in H$. The proof that $\sigma w \sigma w^{-1}$ is again elliptic of order 3 follows at once.

3.2 A and B lie in conjugate H-orbits

Suppose now that $B = \sigma w(V_0)$ for some $\sigma \in H$. Let $h = \sigma w$. Then $h(A) = V_0$, and $h(V_0) = B$. Consider $h^{-1}(A)$. This tube is tangent to $h^{-1}(B) = V_0$ and hence is an element of $\mathcal{OT}(1)$. (Recall the assumption that $OT(2) > 0$.) Hence $h^{-1}(A)$ must be a translate of either A or $w(V_0)$ under H. If $h^{-1}(A) = \alpha(A)$ for some $\alpha \in H$, then the result of the previous section applies since $h^{-1}(A)$ is also tangent to $h^{-1}(V_0) = A$.

Suppose then that $h^{-1}(A) = \alpha w(V_0)$ for some $\alpha \in H$. Then $w^{-1}\sigma^{-1}w^{-1}(V_0) = \alpha w(V_0)$, and hence $w^{-1}\sigma^{-1}w^{-1} = \alpha w \beta$ for some $\beta \in H$. Proceeding as in the previous section, lift everything to $\mathrm{SL}(2, \mathbb{C})$ and assume that α, β, σ are diagonal matrices with diagonal entries (a, a^{-1}), (b, b^{-1}), and (s, s^{-1}) respectively, and that

$$w = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

for some x, y satisfying $x^2 - y^2 = 1$. Then the matrix product $w^{-1}\sigma^{-1}w^{-1} = \alpha w \beta$ expands to

$$\begin{pmatrix} s^{-1}x^2 + sy^2 & -xy(s + s^{-1}) \\ -xy(s + s^{-1}) & sx^2 + s^{-1}y^2 \end{pmatrix} = \begin{pmatrix} abx & ab^{-1}y \\ a^{-1}by & a^{-1}b^{-1}x \end{pmatrix}$$

Multiplying the (1,2) and (2,1) entries on each side,

$$x^2 y^2 (s + s^{-1})^2 = y^2$$

And since $w \neq \pm I$, $y \neq 0$. So $x^2(s + s^{-1})^2 = 1$. But

$$\begin{aligned} \operatorname{tr}(\sigma w) &= sx + s^{-1}x \\ &= x(s + s^{-1}) = \pm 1 \end{aligned}$$

Hence σw is elliptic of order 3, a contradiction. This completes the proof of Lemma 3.2. □

4 A lower bound for the tube radius when $OT(2)$ is small

This section is devoted to proving the following:

Proposition 4.1 *If γ is a geodesic of length l and tuberadius r in a complete orientable hyperbolic 3-manifold, and if $r > 0.15$ and $OT(2) = \rho \leq 0.3$, then*

$$l \geq \frac{1}{\cosh(2r)\sinh(2r)} \left(\frac{{R_{2r,0}}^2}{4} + \left(R_{2r,0.3} - \frac{R_{2r,0}}{2} \right)^2 \right) \tag{4.1}$$

Furthermore if $l \leq 0.103$ then the right-hand side of (4.1) is invertible and provides an implicit lower bound on r as a function of l.

Proof: Choose $V \in \mathcal{OT}(2)$, let v be the center of V, and let v' be the projection of v to the cylinder C. The interior of the hyperbolic ball centered at v of radius $2r$ will not contain the center of any tube in $\mathcal{OT}(1)$. Hence by Lemma 2.4 there is a disk in C centered at v' with radius $R_{2r,\rho}$ whose interior does not contain the center of any tube in $\mathcal{OT}(1)$. We use the following lemma to obtain a radius which is independent of ρ:

Lemma 4.2 *For fixed r and $0 < \rho < 2r$, $R_{2r,\rho}$ is a decreasing function of ρ.*

Proof: From equation (2.6) one can see that because coth is decreasing for positive values and $\cosh^{-1}$ is increasing, it suffices to show that the function

$$\frac{\sinh(2r)\sinh(2r+\rho) + \cosh(2r)}{\cosh(2r)\cosh(2r+\rho)}$$

is decreasing in ρ. But by direct calculation,

$$\begin{aligned} \frac{\partial}{\partial\rho} \frac{\sinh(2r)\sinh(2r+\rho) + \cosh(2r)}{\cosh(2r)\cosh(2r+\rho)} &= \frac{\sinh(2r) - \cosh(2r)\sinh(2r+\rho)}{\cosh(2r)\cosh^2(2r+\rho)} \\ &\leq 0 \end{aligned}$$

because $\cosh(2r)\sinh(2r+\rho) \geq \sinh(2r)$. This proves the lemma. □

Hence the interior of the disk in C centered at v' with radius $R_{2r,0.3}$ does not contain the center of any tube in $\mathcal{OT}(1)$, provided that $r > 0.15$.

Now pack the cylinder C with disks as follows. Around the center of each tube in $\mathcal{OT}(1)$ place a disk of radius $R_{2r,0}/2$ as in section 2. Now in addition, place a disk of radius $R_{2r,0.3} - R_{2r,0}/2$ centered at v'; this disk will not overlap the previous disks. Our goal is to place a disk of radius $R_{2r,0.3} - R_{2r,0}/2$ at each point which is the projection to C of the center of a tube in $\mathcal{OT}(2)$. However, while these "secondary disks" do not overlap the disks of radius $R_{2r,0}$, it has yet to be shown that no two secondary disks overlap each other. The following lemma addresses this point:

Lemma 4.3 *If V_i, $V_j \in \mathcal{OT}(2)$ are tubes with centers v_i, v_j respectively, and if the projections of v_i and v_j to C are the points $v_i{}'$ and $v_j{}'$ respectively, then*

$$d_E(v_i{}', v_j{}') \geq 2R_{2r,0.3} - R_{2r,0} \tag{4.2}$$

Proof: Assume that the cylindrical coordinates of $v_i{}'$ and $v_j{}'$ are $(0,0)$ and (d,ϕ) respectively for some d, ϕ, and use the hyperboloid model of $\mathbb{H}^3$ as in section 2. In particular, assume that γ_0 is the intersection of $\mathbb{H}^3$ with the plane $\{y = z = 0\}$ in $\mathbb{R}^4$, and that v_i, which must lie at a distance of $2r+\rho$ from γ_0, lies at the coordinates

$$(0, 0, \sinh(2r+\rho), \cosh(2r+\rho))$$

Then v_j must lie at the coordinates

$$(\sinh(d)\cosh(2r+\rho), -\sin(\phi)\sinh(2r+\rho), \cos(\phi)\sinh(2r+\rho), \cosh(d)\cosh(2r+\rho))$$

Because $d_H(v_i, v_j) \geq 2r$, equation (2.3) for the metric in the hyperboloid model yields

$$\cosh(d)\cosh^2(2r+\rho) - \cos(\phi)\sinh^2(2r+\rho) \geq \cosh(2r)$$

Then by the same arguments as in the proof of Lemma 2.4, the distance $d_E(v_i{}', v_j{}')$ along C must be at least δ_ρ, where

$$\delta_\rho = \sqrt{\sinh(2r)\cosh(2r)\coth(2r+\rho)}\cosh^{-1}\left(\frac{\sinh^2(2r+\rho)+\cosh(2r)}{\cosh^2(2r+\rho)}\right) \tag{4.3}$$

So it suffices to show that $\delta_\rho \geq 2R_{2r,0.3} - R_{2r,0}$. Now note that coth is decreasing for positive values, $\cosh^{-1}$ is increasing, and

$$\begin{aligned} \frac{\partial}{\partial\rho}\frac{\sinh^2(2r+\rho)+\cosh(2r)}{\cosh^2(2r+\rho)} &= (\cosh^4(2r+\rho))^{-1}\Big(2\sinh(2r+\rho)\cosh(2r+\rho) \\ &\quad -2\cosh(2r)\sinh(2r+\rho)\cosh(2r+\rho)\Big) \\ &\leq 0 \end{aligned}$$

Hence δ_ρ is decreasing in ρ. So it suffices to show that $\delta_{0.3} \geq 2R_{2r,0.3} - R_{2r,0}$. But this inequality is a special case of the following lemma:

Lemma 4.4 *If $0 < \rho < 2r$, then $R_{2r,0} - R_{2r,\rho} \geq R_{2r,\rho} - \delta_\rho$.*

Proof: Let $g(r,v,w) = \cosh^{-1}\left(\frac{\sinh(2r+v)\sinh(2r+w)+\cosh(2r)}{\cosh(2r+v)\cosh(2r+w)}\right)$. Then, after dividing through by $\sqrt{\sinh(2r)\cosh(2r)}$, it must be shown that

$$\begin{gathered}\sqrt{\coth(2r)}\; g(r,0,0) - \sqrt{\coth(2r+\rho)}\; g(r,\rho,0) \geq \\ \sqrt{\coth(2r+\rho)}\; g(r,\rho,0) - \sqrt{\coth(2r+\rho)}\; g(r,\rho,\rho)\end{gathered}$$

We show this by proving that

$$\begin{aligned}\sqrt{\coth(2r)}\; g(r,0,0) - \sqrt{\coth(2r+\rho)}\; g(r,\rho,0) &\geq \\ \sqrt{\coth(2r+\rho/2)}\; g(r,0,0) - \sqrt{\coth(2r+\rho/2)}\; g(r,\rho,0) &\geq \\ \sqrt{\coth(2r+\rho/2)}\; g(r,\rho,0) - \sqrt{\coth(2r+\rho/2)}\; g(r,\rho,\rho) &\geq \\ \sqrt{\coth(2r+\rho)}\; g(r,\rho,0) - \sqrt{\coth(2r+\rho)}\; g(r,\rho,\rho)&\end{aligned}$$

The first and last inequalities are simple consequences of the fact that the hyperbolic cotangent is a decreasing function. The middle inequality is harder to prove. After dividing through by $\sqrt{\coth(2r+\rho/2)}$ we need only show that $g(r,0,0)-g(r,\rho,0) \geq g(r,\rho,0)-g(r,\rho,\rho)$. This will follow from hyperbolic trigonometry applied to certain quadrilaterals with two adjacent right angles (these are sometimes called Saccheri quadrilaterals). The side of such a quadrilateral ending in the two right angles will be called the *base* of the quadrilateral. For convenience, let the four sides in clockwise order be called the west, north, east, and south sides, where the base is the west side.

Using hyperbolic trigonometry (see [F] pg.88), we see that $X = g(r,0,0)$ is the base of the Saccheri quadrilateral with three other sides of length $2r$; $Y = g(r,\rho,0)$ is the base of the Saccheri quadrilateral with three other sides of length $2r+\rho$, $2r$, $2r$ in clockwise order; and $Z = g(r,\rho,\rho)$ is the base of the Saccheri quadrilateral with three other sides of length $2r+\rho$, $2r$, $2r+\rho$ in clockwise order. Note that the two non-base vertices can be thought of as the centers of circles of radius r in all three quadrilaterals, and that in each quadrilateral these circles abut.

Overlap the first and third quadrilaterals in such a way that the base of the third is a subset of the base of the first, and the mid-points of the two bases coincide. Then, the first quadrilateral has excess base length $\frac{X-Z}{2}$ on both ends.

Now, overlap the first and second quadrilaterals, so that the base of the second is a subset of the base of the first, and the west-south vertices coincide. Then, the first quadrilateral has excess base length $X-Y$ on the west-north end. Further, the circle associated to the north-east vertex must dip below the perpendicular bisector to the base of the first quadrilateral. But the north-east-vertex circle for the third quadrilateral does not dip below this line. Hence, $X-Y > \frac{X-Z}{2}$.

This shows that $X-Y \geq Y-Z$, proving the lemma. □

neither simple and appears it is possible to r, say $r \geq 5$. holds for r

By the above two lemmas, C can be packed with disks of two different radii. Specifically, C can be packed with disks of radius $R_{2r,0}/2$ around the centers of tubes in $\mathcal{OT}(1)$, and disks of radius $R_{2r,0.3} - R_{2r,0}$ around those points v' which are the projections to C of centers of tubes in $\mathcal{OT}(2)$. Furthermore, both $\mathcal{OT}(1)$ and $\mathcal{OT}(2)$ consist of two H-orbits by Lemma 1.3. Hence, if A is the area of a fundamental region on C as before, then

$$A \geq 2\pi \left(\frac{R_{2r,0}}{2}\right)^2 + 2\pi \left(R_{2r,0.3} - \frac{R_{2r,0}}{2}\right)^2$$

And as before $A = 2\pi l \cosh(2r)\sinh(2r)$; this and the above equation prove the first part of Proposition 4.1.

4.1 Bounding r in terms of l

As with equation (2.1), the right-hand side of equation (4.1) is not an invertible function. However, comparing the estimate of equation (4.1) with that of equation (2.11), we see that the estimate of equation (2.11) is greater when r is less than $0.23193\ldots$, and lesser otherwise. At this point, both estimates give $l \geq 0.103$. The function on the right-hand side of (4.1) has a single local maximum at approximately $r = 0.591$, and it equals 0.103 when r equals one of $0.2301\ldots$ and $1.2459\ldots$. Thus equation (4.1) and equation (2.11) together determine a lower bound for r in terms of l when $l \leq 0.103$, or equivalently $r \geq 1.246$. This proves the second part of Proposition 4.1. In particular, this combined estimate says that $\pi l \sinh^2(r) \geq 0.95$ when $r \geq 1.483$, and that $r \geq 1.483$ when $l \leq 0.069$.

5 Proof of Theorem 1.1

For any given M, either $OT(2) \geq 0.3$ or $OT(2) \leq 0.3$. In the first case if $r > 0.15$ then Proposition 3.1 applies and says that if $l \leq 0.07$ then $r \geq 1.478$, and consequently $\pi l \sinh(r)^2 \geq 0.9509$. In the second case if $r > 0.15$ then Proposition 4.1 applies and says that if $l \geq 0.069$ then $r \geq 1.483$, and consequently $\pi l \sinh(r)^2 \geq 0.95$.

Additionally, the proof of Proposition 2.1 showed that $r \geq 1.02$ whenever $l \leq 0.10438$, no matter what the magnitude of $OT(2)$. These facts combined together prove Theorem 1.1.

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