ON "THERMODYNAMICS" OF RATIONAL MAPS II. NON-RECURRENT MAPS

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ABSTRACT. The pressure function $p(t)$ of a non-recurrent map is real analytic on some interval $(0, t_*)$ with t_* strictly greater than the dimension of the Julia set. The proof is an adaptation of the well-known tower techniques to the complex dynamics situation. In general, $p(t)$ need not be analytic on the whole positive axis.

1. INTRODUCTION AND RESULTS

1.1. In this paper we study analyticity properties of the *pressure function* of non-recurrent maps. Our approach is based on the well-known *tower techniques* adapted to the complex dynamics situation.

The pressure function $p(t)$, which is defined in terms of the Poincare series, see (1.4) below, carries essential information about ergodic and dimensional properties of the maximal measure. In particular, it characterizes the dimension spectrum of harmonic measure on the Julia set in the case of a polynomial dynamics. According to the classical theory of Sinai, Ruelle, and Bowen, $p(t)$ is real analytic if the dynamics is *hyperbolic*, i.e. expanding on the Julia set. This fact is closely related to the so called "spectral gap" phenomenon which also implies other important features of hyperbolic dynamics such as existence of equilibrium states, exponential decay of correlations, etc. The problem of extending (some parts) of the classical theory to the non-hyperbolic case has become one of the central themes in the ergodic theory of conformal dynamics.

In the first part [8] of this work, we provided a detailed analysis of the negative part $t < 0$ of the pressure function for general rational maps. The case $t > 0$ is substantially more complicated (and more important). The main difficulty comes from the presence of singularities (critical points) on the Julia set. To circumvent this difficulty, we propose to use a tower construction which forces the dynamics to be expanding on some auxiliary space. The tower method has been widely used in the general theory of dynamical systems with some degree of hyperbolicity, see especially [12], and in particular in 1-dimensional real dynamics, where the construction is known as *Hofbauer's tower*. To apply this method in the complex case, it is natural to use some basic elements of the Yoccoz jigsaw puzzle structure, see [9].

We will discuss only the simplest type of non-hyperbolic behavior – the case of *non-recurrent* dynamics, see [2] for a geometric characterization. We will see that the

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tower method provides an almost automatic way to establish real analyticity of the pressure function on some interval $(0, t_*)$ with t_* strictly greater than the dimension of the Julia set. We will also show that it is not always possible to get analyticity on whole positive axis. It might be interesting to apply the tower method to some other classes of conformal dynamical systems, for instance to Collet-Eckmann maps, see [3]. The main problem in the general case is to control the distortion (as in Lemma 1, Section 2 below).

1.2. **Branched Cantor dynamics.** We will consider the following *model* example. where the puzzle piece structure is already explicit. This kind of dynamics appears, for example, if all but one critical points of a polynomial escape to infinity.

Let U_0 be an open Jordan domain, and suppose we have a finite number of open topological discs ("generation one puzzle pieces") P_1 such that their closures $\bar{P}_1$ are disjoint and are contained in U_0. Let

$$F : \bigcup P_1 \to U_0$$

be an analytic function such that the restriction of F to one of the pieces, which we denote by U_1, is a two-fold branched cover $U_1 \to U_0$, and the restriction of F to each of the other P_1's is a conformal isomorphism $P_1 \to U_0$. For each $n \geq 0$, consider the open set

$$F^{-n}U_0 = \{z : \ F^j z \in \bigcup P_1 \quad \text{for} \quad j = 0, 1, \dots, n-1\};$$

its components P_n are called *puzzle pieces* of generation n. The Julia set $J = J(F)$ is defined by the usual Cantor procedure:

$$J = \bigcap_n \bigcup P_n.$$

If $z \in J$, then $P_n(z)$ is the notation for the puzzle piece of generation n containing z. Clearly,

$$F P_n(z) = P_{n-1}(Fz). \tag{1.1}$$

Let c denote the (unique) critical point of F. We will assume $c \in J$, and call the components

$$U_n = P_n(c)$$

critical puzzle pieces. If the dynamics is *non-recurrent*, we can assume without loss of generality that

$$\forall n \geq 1, \qquad c_n := F^n c \notin U_1. \tag{1.2}$$

It is then well known that F is expanding on the postcritical set:

$$\chi_c := \liminf_{n\to\infty} \frac{1}{n} \log |F_n'(c_1)| > 0, \tag{1.3}$$

where, as a general rule, we write F_n' instead of $(F^n)'$.

1.3. **Statement of results.** We define the *pressure function* of F by the formula

$$p(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{z\in F^{-n}c} |F_n'(z)|^{-t}. \tag{1.4}$$

Note that instead of c we can take any other point in U_1.

Theorem A. *Exactly one of the following occurs:*

- $p(t) > -(\chi_c/2)t$ *for all* $t > 0$*, in which case the pressure function is real analytic on* $(0,\infty)$*;*
- *there exists a point* $t_c > \dim J$ *such that* $p(t)$ *is a real analytic function on* $(0, t_c)$*, and* $p(t) \equiv -(\chi_c/2)t$ *on* $[t_c, \infty)$*.*

With some minor changes in the statement and proof, the dichotomy of Theorem A holds for arbitrary non-recurrent rational maps. To avoid unnecessary technical details, we have decided to give the argument only for the model example of generalized conformal Cantor sets. We also believe that the dichotomy should extend to arbitrary Collet-Eckmann maps, i.e. the maps satisfying $p'(\infty) < 0$. We are somewhat less certain about whether the pressure function is always real analytic on $[0, t_0)$ for general rational maps, where t_0 is the first zero of $p(t)$.

The real dynamical version of Theorem A is known for smooth unimodular interval maps satisfying the Collet-Eckmann condition. L.-S. Young [11] and Keller, Nowicki [5] established exponential decay of correlations, (the latter paper is based on Hofbauer's tower), and the paper [1] contains an explicit proof of analyticity of the pressure function for t close to 1.

Theorem B. *For generalized conformal Cantor sets, both alternatives in Theorem A are possible.*

The authors would like to thank Misha Lyubich and Marius Urbanski for useful discussions. Recently, Urbanski [10] gave an alternative proof of Theorem A based on his theory of conformal iteration systems. His ongoing project with Lyubich provides yet another approach, which uses the Lyubich and Minsky construction of hyperbolic laminations [6].

2. Proof of Theorem A

The proof will follow the standard argument applied to the tower dynamics. Without loss of generality, we will assume that the domains U_0 and P_1's have real analytic boundaries.

2.1. **Critical annuli.** For $k \geq 1$, the critical annulus D_k is the domain

$$D_k = U_k \setminus \bar{U}_{k+1}.$$

By assumption (1.2), $F^k : U_k \to U_0$ is a 2-cover branched over c, and so

$$F^k : D_k \to U_0 \setminus \bar{P}_1(c_k)$$

is an unbranched covering map of degree two. If N denotes the total number of non-critical generation one pieces, then there are precisely $2N$ puzzle pieces of generation $k+1$ inside D_k, and F^k maps each of them univalently onto the corresponding piece $P_1 \neq P_1(c_k)$. Let us denote

$$A_n \ = |\ F_n'(c_1)|^{1/2}, \tag{2.1}$$

and

$$Q \ = \ \liminf_{n\to\infty} A_n^{1/n} \ \overset{(1.3)}{=} \ e^{\chi_c/2}. \tag{2.2}$$

Note that $Q > 1$.

Lemma 1. *With constants independent of k, we have*

$$z \in D_k \ \Rightarrow \ |F_k'(z)| \asymp A_k \quad \textit{and} \quad |F_k''(z)| \lesssim A_k^2; \tag{2.3}$$

in particular,

$$|F_k'(z)|^{-2}|F_k''(z)| \lesssim 1. \tag{2.4}$$

Proof: Since we have assumed that the boundaries of U_0 and P_1's are real analytic, the Koebe distortion theorem implies

$$\text{diam } P_{k-1}(c_1) \asymp A_k^{-2},$$

and

$$\forall w \in P_{k-1}(c_1), \quad |F_{k-1}'(w)| \asymp A_k^2.$$

We also have

$$z \in D_k \quad \Rightarrow \quad |z - c| \asymp [\text{diam } P_{k-1}(c_1)]^{1/2} \asymp A_k^{-1}.$$

Applying the chain rule, we get the first relation in (2.3). The second relation then follows from the Cauchy formula. □

2.2. **Tower dynamics.** We will use a version of Hofbauer's construction in one dimensional real dynamics. The tower space $\mathcal{T}$ is defined as a subset of the direct product $U_0 \times \mathbb{N}$:

$$\mathcal{T} = \bigsqcup_{k\geq 1} \mathcal{T}_1,$$

where

$$\mathcal{T}_1 = (\text{union of all } P_1\text{'s}, 1), \qquad \mathcal{T}_k = (U_k, k), \quad (k \geq 2).$$

The *tower map* $T : \mathcal{T} \to \mathcal{T} \cup (U_0, 1)$ acts according to the rule

$$T : (z, k) \mapsto \begin{cases} (z, k+1), & z \in U_{k+1}, \\ (F^k z,\ 1), & z \notin U_{k+1}. \end{cases}$$

Let us introduce a (Riemann) metric on $\mathcal{T}$ as follows. Fix a number $Q_* > 1$ which we call the *tower constant*; this constant will vary in the course of the proof, but we will always assume

$$Q_* < Q. \tag{2.5}$$

We define the metric on the n-th *floor* $\mathcal{T}_n$ so that the natural projection $\mathcal{T}_n \to U_0$ decreases distances by the factor of Q_*^{n-1}. By definition, the distance between any two different floors is equal to infinity. On each floor of the tower, the map T is continuous and differentiable almost everywhere with respect to the Lebesgue measure, namely

$$T'(z,k) = \begin{cases} Q_*, & z \in U_{k+1}, \\ Q_*^{1-k} F_k'(z), & z \notin \bar{U}_{k+1}. \end{cases}$$

By (2.2), (2.3) and (2.5), the tower dynamics is *expanding*:

$$\exists q > 1, \quad \forall n, \quad |T_n'(y)| \geq \text{const } q^n, \tag{2.6}$$

provided that T^n is defined and differentiable at y.

Lemma 2. *The is a constant C independent of n such that if T^n is defined and differentiable at y, then*

$$\frac{|T_n''(y)|}{|T_n'(y)|^2} \;\leq\; C.$$

Proof: The function $|T'|^{-2}|T''|$ is zero if we "go up." If T sends $y = (z,k)$ to the first floor, then $z \in D_k$, and by (2.4) we have

$$\frac{|T''(z,k)|}{|T'(z,k)|^2} \;=\; \frac{|F_k'' z|}{|F_k' z|^2} \;\lesssim\; 1. \tag{2.7}$$

It follows that $|T'|^{-2}|T''|$ is a bounded function. Let X_n denote the supremum of $|T_n'|^{-2}|T_n''|$ on the set where this expression is defined. From the identity

$$\frac{T_n''}{(T_n')^2} = \frac{T_{n-1}'' \circ T}{(T_{n-1}' \circ T)^2} + \frac{T''}{(T')^2}\;\frac{1}{T_{n-1}' \circ T},$$

(differentiate the equation $T^n = T^{n-1} \circ T$ twice), and from (2.6) and (2.7), we have

$$X_n \leq X_{n-1} + Cq^{-n},$$

and so the lemma follows. □

2.3. **Ruelle-Perron-Frobenius transfer operator.** For $t > 0$, the operator L_t is given by the formula

$$L_t f(x) = \sum_{Ty=x} f(y)|T'(y)|^{-t}, \qquad x \in \mathcal{T},$$

Note that the values $T'(y)$ are well defined, and the sum is finite for bounded functions. We will study the action of L_t on the following Banach spaces:

- $C(\mathcal{T})$, the space of bounded continuous functions with the L^∞-norm $\|f\|_\infty$,
- $W^{1,\infty}(\mathcal{T})$, the Sobolev space with the norm $\|f\|_* = \|f\|_\infty + \|\nabla f\|_\infty$.

Note, that the norm in the latter space depends on the choice of Q_*.

Lemma 3. *L_t is a bounded operator in $C(\mathcal{T})$ and in $W^{1,\infty}(\mathcal{T})$.*

Proof: The first statement is clear by construction. The boundedness in $W^{1,\infty}(\mathcal{T})$ follows from Lemma 2 (for more details, see the proof of Lemma 5 below.) □

Define $\lambda(t) = \rho(L_t, C(\mathcal{T}))$, where the latter is the notation for the spectral radius of the operator $L_t : C(\mathcal{T}) \to C(\mathcal{T})$. We want to compare $\lambda(t)$ with the spectral radius $\rho(L_t, W^{1,\infty}(\mathcal{T}))$, and with the *essential* spectral radius

$$\rho_{\text{ess}}(L_t, W^{1,\infty}(\mathcal{T})) = \inf\{\ \rho(L_t - K, W^{1,\infty}(\mathcal{T})) :\ \text{rank } K < \infty\}.$$

Proposition. *If $t > 0$ is such that*

$$p(t) > -(\chi_c/2)t, \qquad (\textit{i.e.}\quad e^{p(t)} > Q^{-t}), \tag{2.8}$$

then there exists a tower constant $Q_* \in (1, Q)$ such that for the corresponding Ruelle-Perron-Frobenius operator L_t we have :

- $\lambda(t) = e^{p(t)}$*;*
- $\rho_{\text{ess}}(L_t, W^{1,\infty}(\mathcal{T})) < \rho(L_t, W^{1,\infty}(\mathcal{T})) = \lambda(t)$*;*
- $\lambda(t)$ *is a simple eigenvalue of L_t in $W^{1,\infty}(\mathcal{T})$:* $\dim\ker(L_t - \lambda(t))^2 = 1$.

Once we have verified this proposition, we can apply the usual argument of analytic perturbation theory to establish real analyticity of $p(t)$ in the interval $(0, t_c)$, where

$$t_c = \inf\{t :\ p(t) = (\chi_c/2)t\}.$$

Theorem A then follows from the general properties of the pressure function. The rest of the section is devoted to the proof of the proposition.

2.4. **Pressure and spectral radius.** Here we prove that $\lambda(t) = e^{p(t)}$ for every $Q_* \in (1, Q)$. To show $\lambda(t) \geq e^{p(t)}$, we first observe that

$$z \in U_1 \quad \Rightarrow \quad T^{-n}(z, 1) \cap \mathcal{T}_1 = (F^{-n}z, 1). \tag{2.9}$$

Indeed, if $F^n w = z$, then T^n sends $(w, 1)$ to the first floor, for if $T^n(w, 1) = (v, k)$ with $k \geq 2$, then $v \in U_k$, and

$$T^{n-k+1}(w, 1) = (v, 1),$$

so $v = F^{n-k+1}w$ and $z = F^{k-1}v \in F^{k-1}U_k = P_1(c_{k-1}) \cap U_1 = \emptyset$.

Let $\mathbf{1}$ denote the constant function $x \mapsto 1$. From (2.9), we have

$$[L_t^n \mathbf{1}](c, 1)\ \geq \sum_{T^n(w,1)=(c,1)} |T_n'(w, 1)|^{-t}\ =\ \sum_{F^n w = c} |F_n' w|^{-t},$$

and therefore

$$\begin{aligned} \log \lambda(t) &= \lim\ \frac{1}{n} \log\ \|L_t^n \mathbf{1}\|_\infty \\ &\geq \limsup\ \frac{1}{n} \log\ [L_t^n \mathbf{1}](c, 1) \\ &\geq \limsup\ \frac{1}{n} \log \sum_{F^n w = c} |F_n' w|^{-t} \geq p(t). \end{aligned}$$

In the proof of the opposite inequality, we will use the assumption (2.8). Let us estimate the value of $L_t^n \mathbf{1}(x)$ at $x = (z,1)$ with $z \in U_1$. If $w \in U_k$, then $T^n(w,k) = (z,1)$ if and only if $F^{n+k-1}w = z$. (Apply (2.9) to the equation $T^{n+k-1}(w,1) = (z,1)$.) In this case we also have

$$|F'_{n+k-1}w| = |T'_{n+k-1}(w,1)| = Q_*^{k-1}|T'_n(w,k)|,$$

and so

$$|T'_n(w,k)| = Q_*^{1-k}|F'_{n+k-1}w|.$$

It follows that

$$\sum_{k=1}^{\infty} \sum_{T^n(w,k)=(z,1)} |T'_n(w,k)|^{-t} \;=\; \sum_{k=1}^{\infty} Q_*^{t(k-1)} \sum_{w\in U_k\cap F^{-(n+k-1)}z} |F'_{n+k-1}(w)|^{-t}. \tag{2.10}$$

By (1.2), every point $w \in U_k \cap F^{-(n+k-1)}z$ belongs to one of the critical annuli D_{k+l} with $l \le n$. Since

$$F^{k+l-1} : D_{k+l} \to U_1 \setminus P_2(c_{k+l-1})$$

is a 2-cover, from $F^{n+k-1} = F^{n-l} \circ F^{k+l-1}$ we derive

$$\begin{aligned}
\sum_{w\in D_{k+l}\cap F^{-(n+k-1)}z} |F'_{n+k-1}(w)|^{-t} &\overset{(2.3)}{\lesssim} A_{k+l-1}^{-t} \sum_{y\in F^{-(n-l)}z} |F'_{n-l}(y)|^{-t} \\
&\lesssim e^{o(n)}\; A_{k+l}^{-t}\; e^{(n-l)p(t)} \\
&\lesssim e^{o(n)}e^{o(k)}\; Q^{-t(k+l)}\; e^{(n-l)p(t)},
\end{aligned}$$

and therefore,

$$\begin{aligned}
\sum_{w\in U_k\cap F^{-(n+k-1)}z} |F'_{n+k-1}(w)|^{-t} &\lesssim e^{o(n)}e^{o(k)}Q^{-tk}e^{np(t)} \sum_{l=1}^{n} Q^{-tl}e^{-lp(t)} \\
&\overset{(2.8)}{\lesssim} e^{o(n)}e^{o(k)}Q^{-tk}e^{np(t)}.
\end{aligned}$$

Returning to the line (2.10), we have

$$[L_t^n\mathbf{1}](z,1) \;\lesssim\; e^{np(t)+o(n)} \sum_{k=1}^{\infty} e^{o(k)}Q_*^{tk}Q^{-tk} \;=\; e^{np(t)+o(n)},$$

and it readily follows that

$$\|L_t^n\mathbf{1}\|_\infty \lesssim e^{nP(t)+o(n)},$$

so $\lambda(t) \le e^{p(t)}$.

2.5. **Two-norm inequality.** Next we establish the Ionescu-Tulcea and Marinescu type inequality for the tower map, see [4]. Fix $t > 0$ satisfying (2.8). We will write L for L_t, λ for $\lambda(t)$, etc.

Lemma 4. *There exist an $o(1)$-sequence and a constant C such that for all integer numbers n and for all $f \in W^{1,\infty}(\mathcal{T})$, we have*

$$\|\nabla(L^n f)\|_\infty \;\le\; o(1)\lambda^n\|f\|_* + C\|\; L^n|f|\;\|_\infty.$$

Proof: We estimate

$$|\nabla(L^n f)(x)| \le I + II,$$

where

$$\begin{aligned} I &\le \sum_{y\in T^{-n}x} |\nabla f(y)|\ |T_n'(y)|^{-t-1} \\ &\le \ \sup_y |T_n'(y)|^{-1}\ \|L^n 1\|_\infty\ \|f\|_* \\ &\overset{(2.6)}{=}\ o(1)\lambda^n \|f\|_*, \end{aligned}$$

and

$$\begin{aligned} II &= \sum_{y\in T^{-n}x} |f(y)|\ |T_n'(y)|^{-t} \frac{|T_n''(y)|}{|T_n'(y)|^2} \\ &\le C\|L^n|f|\|_\infty, \qquad \text{(by Lemma 2).} \end{aligned}$$

□

For an integer $\tau \ge 1$ and for $f \in C(\mathcal{T})$, we denote

$$f_\tau = \begin{cases} f & \text{on} \quad T_1 \cup \cdots \cup \mathcal{T}_\tau, \\ 0 & \text{on} \quad T_{\tau+1} \cup \ldots. \end{cases}$$

Lemma 5. *There are sequences $\tau(n)$ and $C(n)$ such that for all $f \in C(\mathcal{T})$, we have*

$$\|L^n f\|_\infty \le o(1)\lambda^n \|f\|_\infty + C(n)\|f_{\tau(n)}\|_\infty.$$

Proof: Fix $n \gg 1$ and choose a number $\tau = \tau(n) \gg n$ which we will be able to specify at the end of the argument. Denote $h = \mathbf{1} - \mathbf{1}_\tau$. We have

$$\|L^n f\|_\infty \ \le\ \|L^n fh\|_\infty + \|L^n f_\tau\|_\infty \ \le\ \|L^n h\|_\infty\ \|f\|_\infty + \|L^n\|\ \|f_\tau\|_\infty,$$

and so we only need to show $\|L^n h\|_\infty = o(\lambda^n)$. To estimate

$$L^n h(x) = \sum_{T^n y = x,\, \tau(y) > \tau} |T_n'(y)|^{-t},$$

let $x = (z, k)$. We consider two cases.

Case 1: $k > n$. The only point $y \in T^{-n}x$ is $y = (z, k-n)$, and $T_n'(y) = Q_*^n$, so

$$L^n h(x) = Q_*^{-nt} = o(\lambda^n),$$

provided that Q_* has been chosen sufficiently close to Q. (We use here the assumption $\lambda > Q^{-t}$.)

Case 2: $k \le n$. Let $y = (w, m) \in T^{-n}x$ with $m > \tau$. Under n iterations of T, there is at least one drop to the ground floor from level $> \tau$. By (2.6), it follows that $|T_n'(y)| \gtrsim q^\tau$ with $q > 1$, and therefore if $\tau(n) \gg n$, then

$$\sum_{T^n y = x,\, \tau(y) > \tau} |T_n'(y)|^{-t} \ \lesssim q^\tau = o(\lambda^n).$$

□

Corollary. $\|L^n f\|_* \le o(1)\lambda^n \|f\|_* + C(n)\|f_{\tau(n)}\|_\infty$.

In particular, it follows that $\lambda(t)$ is the spectral radius of L_t in $W^{1,\infty}(\mathcal{T})$: iterate the inequality $\|L^N f\|_* \leq (1/2)\lambda^N\|f\|_* + C\|f\|_\infty$, which is true for a sufficiently large N.

2.6. **Quasicompactness.**

Lemma 6. *Fix an interger number $\tau \geq 1$. Given $\delta > 0$, there is a finite rank operator*

$$K : W^{1,\infty}(\mathcal{T}) \to W^{1,\infty}(\mathcal{T})$$

such that

$$\|K\|_* \leq A \qquad \text{with } \ A \geq 1 \ \text{ independent of } \delta, \text{ and} \tag{2.11}$$

$$\forall f, \qquad \|(Kf - f)_\tau\|_\infty \leq \delta\|f_\tau\|_*. \tag{2.12}$$

Proof: Given τ, δ, and f, we define Kf as follows. Set $Kf(\cdot, n) = 0$ for all $n > \tau$. For each $n \leq \tau$, we extend $f(\cdot, n)$ to the whole plane with Sobolev's norm (relative to the n-th floor metric) $\lesssim \|f\|_*$, and consider a grid of equilateral triangles Δ of size δ. We define $Kf(\cdot, n)$ to be a continuous function such that $Kf(z, n) = f(z, n)$ at all vertices z of the grid and such that $Kf(\cdot, n)$ is a linear function in each triangle Δ. The properties (2.11) and (2.12) follow from the construction. □

We can now show that $\rho_{ess}(L_t, W^{1,\infty}(\mathcal{T})) < \lambda$. By the corollary above, we can find n and τ such that the following inequality holds:

$$\|L^n f\|_* \leq \frac{1}{5A}\lambda^n\|f\|_* + C\|f_\tau\|_\infty. \tag{2.13}$$

For this τ, and for $\delta \ll 1$ depending on n and C, we choose a finite rank operator K according to the last lemma. Then

$$\begin{aligned}
\|L^n(I - K)f\|_* &\overset{(2.13)}{\leq} \frac{1}{5A}\lambda^n\|f - Kf\|_* + C\|(f - Kf)_\tau\|_\infty \\
&\leq \frac{2}{5}\lambda^n\|f\|_* + C\delta\|f_\tau\|_* \qquad \text{(by (2.11) and (2.12))} \\
&\leq \frac{3}{5}\lambda^n\|f\|_*,
\end{aligned}$$

and so the essential spectral radius is strictly less than λ.

To compete the proof of Theorem A, we still need to show that $\lambda = \lambda(t)$ is a simple eigenvalue of L_t in $W^{1,\infty}(\mathcal{T})$. The proof of this fact follows from a standard argument, see for example Section 3 of [8], as soon as we can construct a t-*conformal* measure on the tower, i.e. a probability measure ν satisfying

$$L_t^*\nu = \lambda\nu,$$

where L_t^* is the conjugate of $L_t : C(\mathcal{T}) \to C(\mathcal{T})$. The difficulty is of course in the non-compactness of the tower space. The usual construction nevertheless works: we can take for ν any weak*-limit point of the sequence of probability measures ν_n,

$$\nu_n = \frac{\mu_n}{\|\mu_n\|}, \qquad \mu_n := \sum_{k=0}^{n} \lambda^{-k}(L_t^{*k}\delta),$$

where δ is the delta-measure at $(c,1)$. The existence of a weak*-limit point follows from the estimate:

$$\nu_n\left(\bigcup_{j\geq\tau}\mathcal{T}_j\right)=o(1)\quad\text{as } \tau\to\infty \text{ uniformely in } n,$$

which is implicitly contained in the computation in Section 2.4.

3. Proof of Theorem B

3.1. **Sufficient conditions.** We start with two conditions sufficient for the analyticity of the pressure function (the first alternative in Theorem A). For a periodic point $a=F^m a$, let $\chi(a)$ denote its Lyapunov exponent:

$$\chi(a)=\frac{1}{m}\log|F_m'(a)|.$$

Clearly,

$$\inf\{\chi(a):\ a\in\mathrm{Per}(F)\}\ \leq\ \frac{\chi_c}{2}.$$

Proposition 1. *The pressure function is analytic on* $(0,\infty)$ *if*

$$\exists a\in\mathrm{Per}(F),\quad \chi(a)\leq\frac{\chi_c}{2}.$$

Proof: Considering a hyperbolic subset containing a, we have

$$\forall t>0,\quad p(t)>-t\chi(a)\geq-\frac{\chi_c}{2}t,$$

so one can apply Theorem A. □

Proposition 2. *Let* $\{A_n\}$ *and* Q *be as in* (2.1) *and* (2.2). *The pressure function is analytic on* $(0,\infty)$ *if*

$$\forall t>0,\quad \sum\left(\frac{Q^n}{A_n}\right)^t=\infty.$$

We give a proof in Section 3.3 below. This proposition implies that the pressure function is real analytic in the critically finite case, a fact first established in [7]. Using either of the above sufficient conditions, it is easy to construct a critically infinite dynamics without phase transition.

3.2. **Itineraries.** We will need some further notation. For each $w\in J$, we define

$$l(w)=\inf\{l\geq 0:\ F^l w\in U_1\},$$

$$k(w)=k,\qquad\text{if}\quad w\in D_k.$$

Thus $l(w)=0$ iff $w\in U_1$, and $k(w)=0$ iff $w\notin U_1$; $k(w)=\infty$ if $w=c$. Note

$$w\in F^{-m}c\quad\Rightarrow\quad l(w)\leq m,\quad\text{and}\quad k(w)\leq m.$$

To see the latter, assume $w\in P_{m+1}(c)$. Then $c=F^m w\in P_1(c_m)\cap U_{m+1}=\emptyset$, a contradiction.

Let us now fix $n \gg 1$. To each $z \in F^{-n}c$, we assign a finite sequence of integer numbers

$$I(z) = (k_1,\ l_1;\ \dots;\ k_\nu,\ l_\nu)$$

as follows. Starting with $z_1 = z$, we define inductively:

$$k_j = k(z_j), \quad z_j' = F^{k_j} z_j, \quad l_j := l(z_j'),$$

$$z_{j+1} = F^{l_j} z_j' = F^{k_1+l_1+\dots+k_j+l_j} z.$$

We stop as soon as we get $z_{\nu+1} = c$. Clearly, $k_j \geq 1$ for all $j \geq 2$, and

$$k_1 + l_1 + \dots + k_\nu + l_\nu = n. \tag{3.1}$$

3.3. **Proof of Proposition 1.** We will only consider the points with itineraries

$$(k_1, 0; k_2, 0, \dots; k_\nu, 0), \qquad \sum_{j=1}^{\nu} k_j = n, \quad k_j \geq 1.$$

To each such an itinerary there correspond precisely 2^ν points $z \in F^{-n}c$, and by (2.3) we have

$$|F_n'(z)|^{-t} \ \geq \ C^{-t\nu} A_{k_1}^{-t} \dots A_{k_\nu}^{-t} \ = \ Q^{-nt} \alpha_{k_1} \dots \alpha_{k_\nu},$$

where

$$\alpha_{k_j} = \left(\frac{Q^{k_j}}{C A_{k_j}} \right)^t.$$

It follows that

$$\sum_{z \in F^{-n}c} |F_n'(z)|^{-t} \geq Q^{-nt} \sum_\nu 2^\nu c_{n\nu},$$

where

$$c_{n\nu} = \sum_{k_1+\dots+k_\nu=n} \alpha_{k_1} \dots \alpha_{k_\nu}, \tag{3.2}$$

the sum being taken over ordered sequences $(k_1, \dots, k_\nu)$ of positive integers. The result now follows from Theorem A and the following computation:

Let $\{\alpha_k\}$ be a sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \alpha_k > 1, \qquad \limsup_{k \to \infty} \alpha_k^{1/k} = 1.$$

If the numbers $c_{n\nu}$ are defined by (3.2)*, and if $c_n = \sum_\nu c_{n\nu}$, then* $\limsup c_n^{1/n} > 1$.

To see this, we note that the generating function

$$\psi(z) = \sum_{k=1}^{\infty} \alpha_k z^k$$

is analytic in the unit disc. We have

$$\psi^\nu(z) = \sum_n c_{n\nu} z^n,$$

and

$$\frac{1}{1 - \psi(z)} = \sum_\nu \psi^\nu(z) = \sum_n c_n z^n.$$

Since ψ takes the value 1 on the real interval $(0,1)$, the radius of convergence of the latter series is strictly less than one. □

The above argument is almost reversible except that we need to keep track of the constants. This can be done in the following simple example.

3.4. **An example.** We now turn to the second part of Theorem B – we will construct a non-recurrent Cantor dynamics such that the pressure function has a phase transition point.

We start with the unit disc U_0 and two disjoint round discs of radii $\alpha < \beta$ inside $\{|z| < 1/2\}$. On each of two small discs define F to be a linear conformal map onto U_0 (so F maps the centers to 0.) Let $\tilde{J}$ denote the corresponding Cantor set. We choose a point $c_1 \in \tilde{J}$ so that the number of times its orbit visits the disc α in the time interval $[1,n]$ is $\asymp O(\sqrt{n})$. If we denote, as usual, $A_n = |F_n'(c_1)|^{1/2}$, then $Q := \lim A_n^{1/n} = \beta^{-1/2}$, and

$$\sum \frac{Q^n}{A_n} \lesssim \sum \left(\frac{\alpha}{\beta}\right)^{C\sqrt{n}} < \infty. \tag{3.3}$$

Next we add a disjoint critical piece $U_1 = B(c_0, \delta)$ with δ satisfying

$$\delta < 10^{-1}\alpha, \tag{3.4}$$

and define $F : U_1 \to U_0$ by the equation

$$F(z) = B\left(\frac{(z-c_0)^2}{\delta^2}\right), \quad B(w) := \frac{w + c_1}{1 + \bar{c}_1 w}.$$

Note that $c := c_0$ is the critical point and c_1 the critical value.

Proposition 3.

$$\sum_{z \in F^{-n}c} |F_n'(z)|^{-t} \sim Q^{-nt}, \qquad (t \gg 1).$$

It follows that the pressure function $p(t)$ is a linear function for $t \gg 1$, and so the second alternative of Theorem A occurs.

We first establish some lemmas. Denote

$$m_k = \inf_{D_k} |F'|,$$

where D_k are critical annuli, see Section 2.1. With this notation, we have

$$|F_k'(z)| \geq m_k A_{k-1}^2, \qquad (\forall z \in D_k). \tag{3.5}$$

Indeed, F takes D_k to $P_{k-1}(c_1) \setminus P_k(c_1)$, and the map $F^{k-1} : P_{k-1}(c_1) \to U_0$ is just a dilation by A_{k-1}^2 (plus a translation).

Lemma 1. $m_k \geq 2QA_{k-1}^{-1}$.

Proof: If $z \in D_k$, then

$$|Fz - c_1| \ \geq\ \text{dist}(c_1, \partial P_k(c_1)) \\ =\ A_k^{-2}\ \text{dist}(c_{k+1}, \partial U_0)\ \geq\ \frac{1}{2} A_k^{-2}.$$

On the other hand,

$$|Fz - c_1| = |B(w) - B(0)| \leq 2|w|, \qquad w := \frac{\delta^2}{(z-c_0)^2},$$

and therefore

$$|z - c_0| \geq \frac{\delta}{2A_k}.$$

Since $|B'| \geq 1/3$, we have

$$|F'(z)| \geq \frac{2|z-c_0|}{3\delta^2} \geq \frac{1}{3\delta A_k} \\ \geq \frac{\sqrt{\alpha}}{3\delta}\ \frac{1}{A_{k-1}} \geq \frac{2Q}{A_{k-1}},$$

the latter follows by our choice of parameters, see (3.4): $\sqrt{\alpha}/(3\delta) \geq 2/\sqrt{\alpha} > 2Q$. □

We will apply this lemma to estimate the derivative $F_n'(z)$ in terms of the itinerary $I(z) = (k_1,\ l_1;\ \dots;\ k_\nu,\ l_\nu)$ of a point $z \in F^{-n}c$. Using the notation of Section 3.2, we apply the chain rule to the sequence of maps

$$z = z_1 \ \overset{F^{k_1}}{\mapsto}\ z_1' \ \overset{F^{l_1}}{\mapsto}\ z_2 \ \mapsto\ \dots\ \mapsto\ z_\nu \ \overset{F^{k_\nu}}{\mapsto}\ z_\nu' \ \overset{F^{l_\nu}}{\mapsto}\ c.$$

Since the l-maps are all linear, and the estimate (3.5) applies to the k-maps, we get

$$|F_n'(z)|^{-t} \ \leq\ m_{k_1}^{-t}\ A_{k_1-1}^{-2t} \dots m_{k_\nu}^{-t}\ A_{k_\nu-1}^{-2t}\ |F_{l_1}'(z_1')|^{-t} \dots |F_{l_\nu}'(z_\nu')|^{-t}.$$

If we denote $L = l_1 + \dots + l_\nu$ and

$$a_k(t) = 2m_k^{-t} A_{k-1}^{-2t} Q^{kt}, \tag{3.6}$$

then by (3.1) we have

$$|F_n'(z)|^{-t} \ \leq\ Q^{Lt-nt}\ a_{k_1}(t) \dots a_{k_\nu}(t)\ 2^{-\nu}\ |F_{l_1}'(z_1')|^{-t} \dots |F_{l_\nu}'(z_\nu')|^{-t}. \tag{3.7}$$

Lemma 7. *If the functions* $a_k(t)$ *are defined by* (3.6)*, then*

$$\sum a_k(t) \ \to\ 0 \qquad \textit{as} \quad t \to \infty.$$

Proof: By dominating convergence, it is enough to show that the series

$$\sum \frac{Q^k}{m_k A_{k-1}^2}$$

converges and that all terms are $\leq 1/2$. The convergence follows from (3.3) and Lemma 1. By Lemma 1, we also have

$$\frac{Q^k}{m_k A_{k-1}^2} \leq \frac{Q^{k-1}}{2A_{k-1}} \leq \frac{1}{2}.$$

□

Lemma 8. *Denote* $\tilde{p}(t) = \log(\alpha^t + \beta^t)$. *Then*

$$\sum_{I(z)=I} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_\nu}(z'_\nu)|^{-t} \ \le\ 2^\nu e^{L\tilde{p}(t)},$$

where the sum is taken over all points with a given itinerary I.

Proof: Observe first that $\tilde{p}(t)$ is the pressure function of the dynamics $\tilde{F}$ that generates the Cantor set $\tilde{J}$, see Section 3.1, (i.e. $\tilde{F}$ is the restriction of F to the union of the discs α and β). There are 2^n $\tilde{F}$-pieces of generation n, namely $\binom{n}{k}$ discs of radii $\alpha^k\beta^{n-k}$, and so if $z_0 \in U_1$, then

$$\sum_{z\in\tilde{F}^{-n}z_0} |F'_n(z)|^{-t} \ =\ \sum_{k=0}^{n} \binom{n}{k} \alpha^{tk}\beta^{t(n-k)} =\ (\alpha^t + \beta^t)^n = e^{\tilde{p}(t)n}.$$

We can code the points z such that $I(z) = I$ with sequences $[z_1, z'_1, \dots, z_\nu, z'_\nu]$ defined inductively as follows. Start with $z_{\nu+1} := c$ and proceed in the inverse order:

$$z'_j \in \tilde{F}^{-l_j} z_{j+1} \setminus P_1(c_{k_j}), \qquad z_j \in D_{k_j}\ \cap\ F^{-k_j} z'_j.$$

Note that given z'_j and k_j, there are only two choices for z_j. Thus we have

$$\begin{aligned}
\sum_{I(z)=I} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_\nu}(z'_\nu)|^{-t} \ &=\ \sum_{[z_1,z'_1,\dots,z_\nu,z'_\nu]} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_\nu}(z'_\nu)|^{-t} \\
&=\ 2 \sum_{[z'_1,z_2,\dots,z_\nu,z'_\nu]} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_\nu}(z'_\nu)|^{-t} \\
&=\ 2 \sum_{[z_2,z'_2,\dots,z_\nu,z'_\nu]} |F'_{l_2}(z'_2)|^{-t} \dots |F'_{l_\nu}(z'_\nu)|^{-t} \sum_{z'_1\in\tilde{F}^{-l_1}z_2\setminus P_1(c_{k_1})} |F'_{l_1}(z'_1)|^{-t} \\
&\le\ 2e^{l_1\tilde{p}(t)} \sum_{[z_2,z'_2,\dots,z_\nu,z'_\nu]} |F'_{l_2}(z'_2)|^{-t} \dots |F'_{l_\nu}(z'_\nu)|^{-t} \\
&\dots\dots \\
&\le\ 2^\nu e^{(l_1+\dots+l_\nu)\tilde{p}(t)}.
\end{aligned}$$

□

3.5. **Proof of Proposition 3.** Define $\varepsilon = \varepsilon(t) = e^{\tilde{p}(t)}Q^t$, so $\varepsilon < 1$ for large t. We have

$$\begin{aligned}
&\sum_{F^n z=c} |F'_n(z)|^{-t} \qquad \text{by (3.7)} \\
&\le\ Q^{-nt}\sum_\nu 2^{-\nu} \sum_{\nu(I)=\nu} Q^{Lt} a_{k_1}\dots a_{k_\nu} \sum_{I(z)=I} |F'_{l_1}(z'_1)|^{-t} \dots |F'_{l_\nu}(z'_\nu)|^{-t} \qquad \text{by Lemma 3} \\
&\le\ Q^{-nt}\sum_\nu \sum_{\nu(I)=\nu} Q^{Lt} e^{L\tilde{p}(t)} a_{k_1}\dots a_{k_\nu} \\
&=\ Q^{-nt}\sum_\nu \sum_{\nu(I)=\nu} \varepsilon^{l_1}\dots\varepsilon^{l_\nu} a_{k_1}\dots a_{k_\nu}
\end{aligned}$$

Denote

$$c_{n\nu} = \sum_{n(I)=n,\ \nu(I)=\nu} \varepsilon^{l_1} \dots \varepsilon^{l_\nu} \alpha_{k_1} \dots \alpha_{k_\nu}.$$

We need to show that

$$c_n := \sum_\nu c_{n\nu} \lesssim 1.$$

Consider the series

$$\psi(z) = a_1 z + a_2 z^2 + \dots,$$

and

$$\chi(z) = 1 + \varepsilon z + \varepsilon^2 z^2 + \dots = \frac{1}{1 - \varepsilon z}.$$

The series converge in the unit disc. Observe that

$$(1+\psi)\psi^{\nu-1}\chi^\nu = \dots + c_{n\nu} z^n + \dots,$$

because the coefficient of z^n comes from all possible choices for the term $a_{k_1}\varepsilon^{l_1} \dots a_{k_\nu}\varepsilon^{l_\nu} \dots$ with $k_1 + l_1 + \dots + k_\nu + l_\nu = n$; k_1 can be zero. Summing up over ν we obtain the power series $\sum c_n z^n$, and it remains to show that its radius of convergence is at least 1. It's enough to prove this for

$$\sum \varphi^\nu(z) = \frac{1}{1 - \varphi(z)}, \qquad (\varphi := \psi\chi).$$

The latter is true because by Lemma 2,

$$|\varphi(z)| = \frac{1}{1-\varepsilon} \sum a_j < 1, \qquad (|z| < 1),$$

provided t is large enough.

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