ON "THERMODYNAMICS" OF RATIONAL MAPS I. NEGATIVE SPECTRUM

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Abstract. We study the pressure spectrum $P(t)$ of the maximal measure for arbitrary rational maps. We also consider its modified version $\tilde{P}(t)$ which is defined by means of the variational principle with respect to non-atomic invariant measures. It is shown that for negative values of t, the modified spectrum has all major features of the hyperbolic case (analyticity, the existence of a spectral gap for the corresponding transfer operator, rigidity properties etc). The spectrum $P(t)$ can be computed in terms of $\tilde{P}(t)$. Their Legendre transforms are the Hausdorff and the box-counting dimension spectra of the maximal measure respectively. This work is closely related to a paper [32] by D. Ruelle.

1. Introduction and results

This is the first of two papers in which we study the pressure spectrum $P(t)$ of the maximal measure for rational maps, and also some other related parameters. In this part we consider the case $t < 0$.

We begin by briefly introducing the main objects. Let F be a rational map on the Riemann sphere, of degree $d \geq 2$. We write F^n for the n-th iterate of F, and F_n' for the derivative of F^n. Distances and derivatives are measured in the spherical metric. The *Julia set* of F is denoted by J_F. Crit F is the set of all critical points ($\equiv$ zeros of F'), and Per F is the set of periodic points. We refer to [6, 24] for definitions and basic facts of complex dynamics. See also [3, 29, 40] regarding thermodynamical formalism of conformal dynamical systems.

1.1. **Pressure functions.**

Let $\mathcal{M}$ be the set of all F-invariant probability measures on J_F. For $\mu \in \mathcal{M}$, we denote by h_μ the entropy and by χ_μ the Lyapunov exponent of μ, $\chi_\mu := \int \log |F'| \, d\mu$. For each real number t, the corresponding *free energy* of μ is

$$\mathcal{F}_t(\mu) := h_\mu - t\chi_\mu.$$

The *pressure function* (or *spectrum*) $P(t) \equiv P_F(t)$ of F can be defined by means of the so called *variational principle*:

$$P(t) := \sup_{\mu \in \mathcal{M}} \mathcal{F}_t(\mu),$$

see [3, 36]. A measure satisfying $\mathcal{F}_t(\mu) = P(t)$ is called an *equilibrium state* for $\mathcal{F}_t$.

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A version $\tilde{P}(t) \equiv \tilde{P}_F(t)$ of the pressure spectrum is obtained by restricting the class of admissible measures to the subclass $\tilde{\mathcal{M}} \subset \mathcal{M}$ which consists of all invariant *non-atomic* measures:

$$\tilde{P}(t) := \sup_{\mu \in \tilde{\mathcal{M}}} \mathcal{F}_t(\mu).$$

The functions $P(t)$ and $\tilde{P}(t)$ have several useful interpretations, in particular in terms of partition functions or in terms of certain transfer operators. Some of these approaches are mentioned below. Our main result concerning the behavior of the pressure functions for $t < 0$ is the following. Denote

$$\chi_{\max} := \sup\{\chi_a : \ a \in \text{Per } F\},$$

where χ_a is the Lyapunov exponent of a periodic point $\{a\}$:

$$\chi_a := \frac{1}{n} \log |F_n'(a)|, \qquad (n = \text{period of } a).$$

Theorem A. *For an arbitrary rational map F, the pressure function $\tilde{P}_F(t)$ is real analytic on $(-\infty, 0)$, and*

$$P_F(t) = \max\{\tilde{P}_F(t), -\chi_{\max} t\}.$$

1.2. **Transfer operators.** Our motivation for Theorem A comes from the classical theory of Sinai, Ruelle, and Bowen (see [3, 29, 30, 31, 34]) which applies to the (*"hyperbolic"*) case where the dynamics is *expanding* on the Julia set (i.e. $||F'|| > 1$ on J_F with respect to a smooth conformal metric defined near J_F), and from a more recent paper [32] by Ruelle.

Let L_t denote the (Ruelle-Perron-Frobenius) *transfer operator* which acts in appropriate function spaces according to the formula

$$L_t f(z) = \sum_{y \in F^{-1}(z)} f(y) \ |F'(y)|^{-t} \tag{1.1}$$

(the preimages are counted with multiplicities). Let us recall some properties of the hyperbolic case.

- The operators L_t act in the space $C(J_F)$ of continuous functions and $\lambda(t) := e^{P(t)}$ is the spectral radius of L_t.
- $\lambda(t)$ is a simple eigenvalue of L_t. There is a strictly positive eigenfunction f_t, and there is a unique probability measure ν_t on J_F which is an eigenvector of the adjoint operator.
- The probability measure $\mu_t := f_t \ \nu_t$ (we always assume $\nu_t(f_t) := \int f_t d\nu_t = 1$) is a unique equilibrium state for the free energy $\mathcal{F}_t$. We also have the usual form of the *Perron-Frobenius theorem*:

$$\lambda(t)^{-n} \ L_t^n \ \varphi \ \to \ \nu_t(\varphi) \ f_t \quad \text{as} \quad n \to \infty, \quad (\forall \varphi \in C(J_F)). \tag{1.2}$$

- The operator L_t is *quasicompact* in the space $\mathcal{H}_\alpha$ of Hölder continuous (with exponent $\alpha > 0$) functions on J_F. Quasicompactness means that the essential spectral radius of $L_t : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is strictly smaller than the spectral radius $\lambda(t)$. Moreover, the eigenvalue $\lambda(t)$ has spectral multiplicity one, and there are no other eigenvalues of the same modulus.

- Consequently, the convergence in (1.2) is exponentially fast for Hölder continuous functions, and the pressure function $P(t)$ is real analytic.

The same is true for transfer operator of the form

$$Lf(z) \equiv L_g f(z) = \sum_{y \in F^{-1}(z)} f(y)\ g(y) \tag{1.3}$$

provided that F is expanding and the *weight function* g is Hölder continuos and *strictly* positive.

In [32], Ruelle extended some of the mentioned properties to transfer operators associated with arbitrary, *non*-expanding rational maps F assuming that the weight function $g \geq 0$ belongs to the space BV_2 ($\equiv$ functions for which the second derivatives are complex measures) and satisfies a certain integrability condition at all critical points of F. (The weights $g = |F'|^{-t}$, corresponding to our operators L_t, satisfy this condition for every $t < 0$.) He showed that in this case

L is quasicompact in BV_2 provided that g satisfies the following additional condition

$$\exists n : \quad \lambda^n > \sup_{J_F}\ g_n, \tag{1.4}$$

where λ is the spectral radius of L in $C(J_F)$, and $g_n := \prod_{j=0}^{n-1}\ g \circ F^j$. Moreover, λ is an eigenvalue of L and there is a non-negative eigegenfunction.

To relate the spectral radius λ of L to the pressure

$$P(\log g) := \sup_{\mu \in \mathcal{M}}\ [h_\mu + \mu(\log g)],$$

Ruelle referred to the following fact which is due to Przytycki [28]:

Let F be an arbitrary rational function and let g be a non-negative continuous function on J_F. If λ denotes the spectral radius of the transfer operator (1.3) in $C(J_F)$, then

$$\log \lambda = P(\log g).$$

The quasicompactness of transfer operators in spaces of smooth functions is usually derived from the *smoothness improving property* of the operators $\lambda^{-1}L$. In the hyperbolic case, this property follows from the expanding nature of the dynamics. To establish quasicompactness in the non-hyperbolic case, one can try to find an appropriate functional space which relates to the smoothness "on the average" or in some other generalized sense. Ruelle's choice of BV_2 seems to have been motivated by the similarity with the space BV (functions of bounded variation) which is widely used in one-dimensional real dynamics.

In this paper we will use the Sobolev spaces $W_{1,p}$. They work almost as well as the space BV_2, but the corresponding estimates are somewhat simpler. We state a version of Ruelle's theorem for Sobolev spaces in Section 2.7. Moreover, our approach gives a weaker condition (cf. (2.6)) than the condition (1.4) in Ruelle's theorem. This weaker condition (unlike the latter) is always satisfied for the weights $g = |F'|^{-t}$ with $t < 0$, so the operators L_t are always quasicompact in appropriate Sobolev spaces. On the other hand, the well-known (see, e.g., [14, 7, 28, 13]) condition (1.4) will be used for uniqueness of the equilibrium states. We discuss the uniqueness problem in the next subsection.

1.3. **Exceptional maps and "phase transition".** The function $P_F(t)$ may or may not be real analytic. We will see that the first possibility is more typical. We say that F has a *phase transition* if $P_F(t)$ is *not* real analytic on $(-\infty, 0)$. (See the papers [2, 10, 18, 25] for the "physical" interpretation of this phenomenon.) If this is the case, then we have a phase transition point $t_c < 0$ at which the first derivative has a jump discontinuity, and

$$P_F(t) = \begin{cases} \tilde{P}_F(t), & t_c \le t \le 0, \\ -\chi_{\max}\, t, & -\infty < t \le t_c. \end{cases}$$

To characterize the phase transition case, we need the following definition. A rational map F is said to be *exceptional* if there is a finite, non-empty set Σ such that

$$F^{-1}\Sigma \ \setminus \text{Crit } F \ = \ \Sigma. \tag{1.5}$$

Any such Σ has at most four elements (at most two in the polynomial case, cf. [21]), and so there is a maximal set $\Sigma \equiv \Sigma_F$ satisfying (1.5). This set contains at least one periodic orbit, and we define

$$\chi_* := \max \{\chi_a : \ a \in \Sigma_F \ \cap \ \text{Per } F\}.$$

Theorem B. *A rational map F has a phase transition if and only if F is exceptional and*

$$\chi_* \ > \ \sup \{\chi_\mu : \ \mu \in \mathcal{M}, \ \mu(\Sigma_F) = 0\}. \tag{1.6}$$

According to an unpublished result by F. Przytycki, the supremum of Lyapunov exponents in (1.6) can be computed by considering only periodic cycles and so (1.6) is equivalent to the condition

$$\chi_* \ > \ \sup \{\chi_a : \ a \in \text{Per} F \setminus \Sigma_F\}. \tag{1.7}$$

On the other hand, it is easy to see that if the exponent of a periodic point can not be approximated by the exponents of periodic cycles with arbitrarily large periods, then this point has to be in the exceptional set. Thus a rational map has a phase transition if and only if there is a finite number of periodic points such that their Lyapunov exponents are larger than the exponents of all other periodic points by a positive constant.

The algebraic condition (1.5) means that the (local) geometry of J_F near Σ_F is different from the geometry of other parts of the Julia set. The meaning of (1.6) or (1.7) in the polynomial case is the following: the Julia set has a "tip" at some point of Σ_F, and this tip is substantially more "pointed" than any tip in $J_F \setminus \Sigma_F$, see [21].

In terms of equilibrium distributions, one can describe the phase transition case as follows. For each $t \in (t_c, 0)$ there is a unique equilibrium state which is supported by the whole Julia set. At $t = t_c$ we have another equilibrium state that lives on a periodic cycle in Σ_F and persists for $t < t_c$. The original equilibrium state, however, extends analytically to $\{t < t_c\}$ but its free energy $\tilde{P}(t)$ is now smaller than that of the new (degenerate) state. Thus we can think of $\tilde{P}(t)$ as a "hidden" pressure spectrum which can be obtained by the analytic continuation of $P(t)$. Note that this phenomenon differs from the phase transition in the positive spectrum (e.g. for parabolic maps, see [9]), or from the one described in [10].

Critically finite rational functions with parabolic orbifolds (see, e.g., [8, 24]) provide important examples of exceptional maps. Recall that if F is critically finite, i.e. if

$$\#\left\{\bigcup_{n\geq 0} F^n(\text{Crit } F)\right\} \;<\; \infty,$$

then the *ramification function*

$$\nu \equiv \nu_F : \hat{\mathbb{C}} \to \mathbb{N} \cup \{\infty\}$$

can be defined as a minimal function satisfying the following condition:

$$\nu(Fx) \quad \text{is a multiple of} \quad \nu(x)\ \deg_x F, \qquad (\forall x \in \hat{\mathbb{C}}).$$

The *orbifold* $(\hat{\mathbb{C}}, \nu_F)$ is *parabolic* if its Euler characteristic

$$2 - \sum_{x\in\hat{\mathbb{C}}} \left(1 - \frac{1}{\nu(x)}\right)$$

is zero, or, equivalently, if

$$\nu(Fx) = \ \nu(x)\ \deg_x F, \qquad (\forall x \in \hat{\mathbb{C}}).$$

The latter implies that the set $\Sigma = \{x :\ \nu(x) = \max\ \nu\}$ satisfies (1.5), and therefore maps with parabolic orbifolds are exceptional.

The Euler characteristic of an orbifold $(\hat{\mathbb{C}}, \nu_F)$ is zero if and only if the set of values of ν at the ramification points is one of the following:

$$(2,2,2,2),\ (3,3,3),\ (2,4,4),\ (2,3,6),\ (2,2,\infty), \quad \text{or } (\infty,\infty).$$

The latter two cases correspond to Chebychev's polynomials and to the maps $z^{\pm d}$ respectively. In the four former cases, the Julia set is the whole Riemann sphere. One can show that

$$P(t) = \max\{1-t, -2t\}\ \log d$$

for Chebychev's polynomials, and

$$P(t) = \max\{1 - \frac{t}{2}, -kt\}\ \log d \quad \text{with} \quad k = 1, \frac{3}{2}, 2, 3$$

for the types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$ respectively. These maps play a special role in many questions of rational dynamics. The fact that is relevant to our study is essentially due to Zdunik [38]:

Theorem C. *Let F be a rational function. Then $\tilde{P}''_F(t) = 0$ for some/every point $t < 0$ if and only if the function F is critically finite and the corresponding orbifold is parabolic.*

It is easy to give examples of exceptional maps other than critically finite. For instance, the family

$$F_\lambda(z) = \frac{(z+\lambda)^2}{z}, \qquad (\lambda \in \mathbb{C},\ \lambda \neq 0),$$

consists of exceptional maps with $\Sigma_F = \{0, \infty\}$. See also polynomial examples in [21]. Note though that polynomials with $\#\Sigma_F = 2$, and rational functions with $\#\Sigma_F = 4$ must be critically finite of types $(2, 2, \infty)$ and $(2, 2, 2, 2)$ respectively.

1.4. **Dimension spectrum of the maximal measure.** Our results concerning the pressure functions $P(t)$ and $\tilde{P}(t)$ can be interpreted in terms of the *"multifractal analysis"* of the maximal measure. See the book [27] for background material.

Recall that for every rational map F, there is a natural invariant measure m,

$$m := \text{weak*-}\lim_{n\to\infty} \frac{1}{d^n} \sum_{y \in F^{-n}(z)} \delta_y,$$

where d is the degree of F and z is any complex number (with at most two exceptions). The measure m is called the *maximal measure* of F. It was characterized in [4, 11, 19] as a unique invariant probability measure with entropy equal to $\log d$.

If F is a polynomial, then m is the *harmonic measure* of J_F evaluated at infinity. Harmonic measure is a basic object of harmonic and complex analysis, and there is extensive literature relating the properties of harmonic measure to the geometry of the boundary.

To each point $z \in J \equiv J_F$ one can associate a range of local dimensions α given by

$$\liminf_{r\to 0} \frac{\log\ m(B(z,r))}{\log\ r} \ \le\ \alpha\ \le \limsup_{r\to 0} \frac{\log\ m(B(z,r))}{\log\ r},$$

where $B(z, r)$ is the ball of radius r about z and lim inf and lim sup are called the upper and lower pointwise dimensions of m at z respectively. For many z these pointwise dimensions will be equal so that one can talk of a *local dimension* $\alpha(z)$. Typically, there are large fluctuations in the value of $\alpha(z)$ as z ranges over J. The multifractal analysis is a description of the fine-scale geometry of the set J whose "components" are the subsets $\{z :\ \alpha(z) = \alpha\}$ with a homogenous concentration of m parameterized by $\alpha \in \mathbb{R}$.

The *Hausdorff dimension spectrum* of m is defined as the function

$$\tilde{f}(\alpha) := \dim\ \{\alpha(z) = \alpha\},$$

where dim denotes the Hausdorff dimension. Note that the standard notation for the Hausdorff spectrum is $f(\alpha)$ but we reserve the latter for the definition based on box-counting methods (which are supposed to be more relevant for numerical simulations). Namely, we define the *box-dimension spectrum* $f(\alpha)$ of m as the limit (assuming its existence)

$$f(\alpha) := \lim_{\delta\to 0} \frac{\log\ N(\delta, \alpha)}{|\log\ \delta|},$$

where $N(\delta, \alpha)$ denotes the number of squares Q of a δ-grid satisfying $m(Q) \approx \delta^\alpha$; see Section 5 for an accurate definition.

To relate the dimension spectra to the pressure functions, we denote

$$s(t) := \frac{P(t)}{\log d} \quad \text{and} \quad \tilde{s}(t) := \frac{\tilde{P}(t)}{\log d}.$$

It is well-known that if F is *hyperbolic*, then $f(\cdot) \equiv \tilde{f}(\cdot)$ and $s(\cdot) \equiv \tilde{s}(\cdot)$, and these functions are Legendre-type transformations of each other:

$$s(t) = \sup_{\alpha} \frac{f(\alpha) - t}{\alpha}, \tag{1.8}$$

$$f(\alpha) = \inf_{t} [t + \alpha s(t)]. \tag{1.9}$$

If, in addition, F is not conjugate to $z^{\pm d}$, then the equation

$$\alpha s'(t) = -1$$

establishes a one-to-one correspondence between the points $t \in \mathbb{R}$ and the local dimensions α in some interval $(\alpha_{\min}, \alpha_{\max})$. In particular, the negative t-axis corresponds to the interval $(\alpha_{\min}, \alpha_0)$, where

$$\alpha_0 = |s'(0)|^{-1} = \dim m.$$

Here $\dim m$ denotes the Hausdorff dimension of the measure m, i.e. the maximal Hausdorff dimension of a Borel set supporting m.

For general, *non-hyperbolic* rational maps, we have the following result.

Theorem D. *Suppose F is not a critically finite map with parabolic orbifold. Denote $\alpha_0 := \dim m$. Then*

(i) the functions $s(t)$ on $\{t \leq 0\}$, and the function $f(\alpha)$ on $\{\alpha \leq \alpha_0\}$ form a Legendre pair in the sense of (1.8)-(1.9)*;*

(ii) the same is true for $\tilde{s}(t)$, $t \leq 0$, and $\tilde{f}(\alpha)$, $\alpha \leq \alpha_0$.

This theorem shows that the Hausdorff dimension spectrum always has hyperbolic-type behavior: if we set $\tilde{\alpha}_{\min} := \sup\ \{\alpha : \ \tilde{f}(\alpha) > -\infty\}$, then

$$\tilde{f}(\alpha) \quad \textit{is real analytic on the interval} \quad (\tilde{\alpha}_{\min}, \alpha_0).$$

On the other hand, the box-counting spectrum may have a discontinuity in the *second* derivative.

1.5.
The paper is organized as follows. In Section 2, we prove quasicompactness of the transfer operators L_t in appropriate Sobolev spaces. In Section 3, we establish analyticity of the pressure function assuming the existence of a non-atomic eigenmeasure. In Section 4, we study the phase transition case and complete the proofs of Theorems A and B. Theorem C is discussed in Section 3.8. Finally, in Section 5, we study the dimension spectrum of the maximal measure and prove Theorem D.

In what follows, we consider only the polynomial case. This allows to replace some of the dynamical arguments with shorter proofs based on complex analysis, and also to reduce the number of cases in the study of exceptional maps. There is no difficulty in extending the proofs to general rational maps.

In the study of the pressure spectrum, the case $t \geq 0$ is considerably more difficult than the case $t < 0$. We have only partial results concerning the positive part of the pressure spectrum for some special classes of polynomials. This will be the topic of the second part [22] of our work.

For related recent results and further references see [1, 5, 7, 12, 15, 17, 28, 33, 37].

2. Transfer operators in Sobolev spaces

In this section we prove the quasicompactness of the operators L_t (see (1.1)) in appropriate Sobolev spaces. The proof is based on the standard technique – the two-norm inequality of Ionescu-Tulcea and Marinescu [16]. To state the result, we introduce the following notation.

Let F be a polynomial of degree d. Fix a large open disc Ω containing J_F such that

$$F^{-1}\Omega \subset \Omega.$$

For technical reasons we always assume that *the orbits of critical points of F do not intersect the boundary* $\partial\Omega$. We will consider the operators L_t in $C(\bar{\Omega})$ and in the Sobolev spaces $W_{1,p}(\Omega)$. We write $\rho(L_t, X)$ for the *spectral radius* of L_t in the corresponding functional space X,

$$\rho := \lim_{n\to\infty} \|L_t^n\|_X^{\frac{1}{n}}.$$

The *essential spectral radius* is denoted by

$$\rho_{\mathrm{ess}}(L_t, X) := \inf\,\{\rho(L_t - K, X) :\ K \text{ compact operator in } X\}.$$

2.1.

Theorem. *Let $t < 0$. Then for all $p > 2$ sufficiently close to 2, the transfer operator L_t is bounded in $W_{1,p}(\Omega)$, and*

$$\rho_{ess}(L_t, W_{1,p}(\Omega)) \,<\, \rho(L_t, W_{1,p}(\Omega)) \,=\, \rho(L_t, C(\bar{\Omega})).$$

The proof takes the rest of this section. We begin by recalling some properties of $W_{1,p}(\Omega)$. See [39] for general reference.

2.2. **Sobolev spaces.** The Sobolev space $W_{1,p}(\Omega)$, $p \geq 1$, is equipped with the norm

$$\|f\|_{1,p} := \|f\|_p + \|\nabla f\|_p,$$

where $\|\cdot\|_p$ is the L^p-norm. We will need only the case $p > 2$. It is well known that for $p > 2$, the elements of $W_{1,p}(\Omega)$ can be represented as continuous functions and the embedding

$$W_{1,p}(\Omega) \subset C(\bar{\Omega})$$

is a compact operator. Moreover, continuous $W_{1,p}$-functions are *Hölder continuous*:

$$(\,|x-y| \leq \delta\,) \quad \Rightarrow \quad \Big(\,|f(x)-f(y)| \lesssim \delta^{1-\frac{2}{p}} \|\nabla f\|_{L^p(B(x,\delta))}\Big)\,. \tag{2.1}$$

The embedding result will be used in the following form.

Lemma. *There is a constant C (depending on Ω and on $p > 2$) such that for any $\varepsilon > 0$ there exists a finite rank operator K in $W_{1,p}(\Omega)$ such that*

$$\begin{aligned} \|K\|_{1,p} \;&\leq\; C, \\ \|f - Kf\|_\infty \;&\leq\; \varepsilon\, \|f\|_{1,p}. \end{aligned} \tag{2.2}$$

Proof: Extend f to the whole plane with Sobolev norm $\asymp \|f\|_{1,p}$, and consider a grid of equilateral triangles Δ of size $\delta \ll 1$. Define Kf to be a continuous function satisfying

$$Kf = \begin{cases} f & \text{at all vertices} \\ \text{linear in each triangle } \Delta. \end{cases}$$

Then for each Δ, we have the following estimates:

$$|\nabla(Kf)| \lesssim \frac{1}{\delta}\|f - f(\text{center})\|_{L^\infty(\Delta)} \overset{(2.1)}{\lesssim} \delta^{-\frac{2}{p}} \left(\int_{\Delta^*} |\nabla f|^p\right)^{\frac{1}{p}},$$

where Δ^* is the union of Δ with the adjacent triangles. It follows that

$$\int_\Delta |\nabla Kf|^p \lesssim \int_{\Delta^*} |\nabla f|^p.$$

Summing up over all Δ's, we obtain the first inequality. The second inequality follows from (2.1) by the choice of δ. □

2.3.

Lemma. *If $p > 2$ and $t < -2(1 - \frac{2}{p})$, then $L_t\ W_{1,p}(\Omega) \subset W_{1,p}(\Omega)$.*

Proof: Let $f \in W_{1,p}(\Omega)$. Changing the variable in the integral we obtain

$$\int_\Omega |\nabla(Lf)|^p \leq \int_\Omega |\nabla(f|F'|^{-t})|^p\ |F'|^{2-p} \lesssim I + II,$$

where

$$I := \int_\Omega |\nabla f|^p |F'|^{-tp+2-p} \leq \text{const}\ \|f\|_{1,p},$$

(because $-tp + 2 - p > 0$), and

$$II := \int_\Omega |f|^p\ |\nabla(|F'|^{-t})|^p |F'|^{2-p} \leq \|f\|_\infty^p \int_\Omega |\nabla(|F'|^{-t})|^p\ |F'|^{2-p}.$$

To see that the latter integral is finite, we only need to consider neighborhoods of critical points. Suppose c is a zero of F' of order $k \geq 1$. Then we have (as $z \to c$):

$$|\nabla(|F'|^{-t}) \lesssim |z|^{-1-kt},$$

and

$$|\nabla(|F'|^{-t})|^p\ |F'|^{2-p} \lesssim |z|^{-p(1+kt)+k(2-p)}.$$

Since the inequality $t < -(1 + \frac{1}{k})(1 - \frac{2}{p})$ implies

$$-p(1 + kt) + k(2 - p) > -2,$$

the integral converges. □

2.4. **The function s(t).** We define

$$s(t) := \log_d \rho(L_t, C(\bar\Omega)).$$

We will see later (Remark 3.4) that $\rho(L_t, C(\bar\Omega)) = \rho(L_t, J_F)$, and therefore by Przytycki's result (see Section 1.2), we have

$$s(t) = \frac{P(t)}{\log d},$$

in agreement with notation in Introduction. Some preliminary properties of $s(t)$ are stated in the following lemma.

Lemma. *(i) For every point $z_0 \in \partial\Omega$, we have*

$$L_t^n 1(z_0) \asymp \|L_t^n\|_\infty,$$

and therefore

$$\sum_{y\in F^{-n}(z_0)} |F_n'(y)|^{-t} \;=\; d^{s(t)n+o(n)}.$$

(ii) The function $s(t)$, $t \le 0$ is strictly decreasing and satisfies the inequality

$$s(kt) \le k\, s(t) \qquad (\forall k \ge 1). \tag{2.3}$$

Proof: Since $t \le 0$, the function $z \mapsto L_t^n\, 1(z)$ is subharmonic, and therefore we have

$$\|L_t^n\|_\infty \;=\; \|L_t^n\, 1\|_\infty \;=\; \sup_{\partial\Omega} L_t^n 1\ .$$

If $z_1,\ z_2 \in \partial\Omega$, then we can choose a simply connected domain that contains z_1 and z_2 but does not contain forward iterates of the critical points. All branches of F^{-n} are conformal on such a domain, and by the distortion theorem we have

$$|F_n'(y_1)| \asymp |F_n'(y_2)| \quad \text{as} \quad n \to \infty,$$

where $y_1,\ y_2$ denote the images of $z_1,\ z_2$ under the same branch. It is easy to see that the constants in this relation can be chosen independent of the points $z_1,\ z_2$. This completes the proof of the first statement.

Similar argument and the area estimate show that

$$\sum_{y\in F^{-n}(z_0)} |F_n'(y)|^{-2} \lesssim 1\ .$$

By Hölder's inequality,

$$d^n = \left(\sum_{y\in F^{-n}(z_0)} 1\right) = \left(\sum_{y\in F^{-n}(z_0)} |F_n'(y)|^{-t}\right)^{\frac{2}{2-t}} \left(\sum_{y\in F^{-n}(z_0)} |F_n'(y)|^{-2}\right)^{\frac{-t}{2-t}},$$

and we have $s(t) \ge 1 - \frac{t}{2}$, and $s'(0-) \le -\frac{1}{2}$, so $s(t)$ is strictly decreasing. To prove (2.3), we simply observe that $L_{kt}^n\, 1 \le (L_t^n\, 1)^k$. □

2.5. **Two-norm inequality.**

Lemma. *Let t and p be as in Lemma 2.3. Then there exists a positive number $\varepsilon = \varepsilon(p,t)$ such that*

$$\|L_t^n f\|_{1,p} \le d^{n(s(t)-\varepsilon)+o(n)}\|f\|_{1,p} + C_n\|f\|_\infty, \quad (f \in W_{1,p}(\Omega)). \tag{2.4}$$

Proof: We have

$$\begin{aligned}\int_\Omega |\nabla(L_t^n f)|^p &\lesssim \int_\Omega \left(\sum_{y\in F^{-n}(z)} |(\nabla f)(y)|\, |F_n'(y)|^{-(1+t)}\right)^p dA(z)\\ &+ \int_\Omega \left(\sum_{y\in F^{-n}(z)} |f(y)|\, |F_n'(y)|^{-1}\, |\nabla(|F_n'|^{-t})(y)|\right)^p dA(z)\\ &:= I + II\,.\end{aligned}$$

By the argument of the previous lemma, we have

$$II \le C_n^p\, \|f\|_\infty^p.$$

On the other hand, by Hölder's inequality, we have

$$I \le \int_\Omega \left(\sum_{y\in F^{-n}(z)} |(\nabla f)(y)|^p\, |F_n'(y)|^{-2}\right)\left(\sum_{y\in F^{-n}(z)} |F_n'(y)|^{p'(\frac{2}{p}-1-t)}\right)^{\frac{p}{p'}} dA(z),$$

where p' is the conjugate exponent (i.e. $p^{-1} + (p')^{-1} = 1$). Using the obvious relation $|F_n'(y)|^{-2}dA(z) = dA(y)$, we obtain the estimate

$$I \le \|\nabla f\|_p^p \left\|L^n_{p'(1+t-\frac{2}{p})}\right\|_\infty^{\frac{p}{p'}}.$$

It remains to note that

$$\left\|L^n_{p'(1+t-\frac{2}{p})}\right\|_\infty^{\frac{1}{p'}} = d^{n\frac{1}{p'}s(p'(1+t-\frac{2}{p}))+o(n)},$$

and that

$$\frac{1}{p'}s\left(p'\cdot\left(1+t-\frac{2}{p}\right)\right) < \frac{1}{p'}s(p't) \le s(t)$$

by Lemma 2.4. □

2.6. **Proof of Theorem.** Fix numbers $t<0$ and $p>2$ satisfying

$$t < -2\left(1-\frac{2}{p}\right).$$

The transfer operator L_t is bounded in $W_{1,p}(\Omega)$ by Lemma 2.3. By Lemma 2.5, for any given $q \in (0,1)$, we can find an integer N and a constant Q such that

$$\|L_t^N f\|_{1,p} \le q\, d^{Ns}\|f\|_{1,p} + Q\|f\|_\infty, \quad (s := s(t)). \tag{2.5}$$

By induction, we have

$$\|L_t^{kN} f\|_{1,p} \le d^{kNs}\|f\|_{1,p} + QM_k\|f\|_\infty, \quad (k=1,2,\dots),$$

where the sequence $\{M_k\}$ is determined by the equations

$$M_1 = 1, \quad M_{k+1} = d^{Ns}M_k + \|L_t^{kN}\|_\infty.$$

Since d^s is the spectral radius of L_t in $C(\bar{\Omega})$, we have

$$M_k \leq d^{(k+o(k))Ns} \quad \text{as} \quad k \to \infty.$$

It follows that

$$\|L_t^{kN}\|_{1,p} \leq d^{(k+o(k))Ns},$$

and

$$\rho(L_t, W_{1,p}(\Omega)) \leq d^s.$$

The opposite inequality is obvious:

$$\|L_t^n\|_\infty = \|L_t^n 1\|_\infty \lesssim \|L_t^n 1\|_{1,p} \lesssim \|L_t^n\|_{1,p}.$$

Let us now prove the strict inequality for the essential spectral radius. The argument is again based on the estimate (2.5), in which we choose q such that

$$q \ < \ \frac{1}{3(1+C)},$$

where C is the constant in Lemma 2.2. We also take

$$\varepsilon \ < \ \frac{1}{3Q} \ d^{Ns}$$

in (2.2). By Lemma 2.2, there is a finite rank operator K satisfying

$$\|K\|_{1,p} \leq C, \quad \|f - Kf\|_\infty \leq \varepsilon \ \|f\|_{1,p}.$$

Thus we have

$$\begin{aligned} \|L_t^N(f - Kf)\|_{1,p} \ &\leq \ q d^{Ns} \|f - Kf\|_{1,p} + Q\|f - Kf\|_\infty \\ &\leq \ d^{Ns} q(1+C) \ \|f\|_{1,p} + Q\varepsilon \ \|f\|_{1,p} \\ &\leq \ \frac{2}{3} d^{Ns} \ \|f\|_{1,p}, \end{aligned}$$

and therefore

$$\rho_{\text{ess}}(L_t, W_{1,p}(\Omega)) \leq \left(\frac{2}{3}\right)^{\frac{1}{N}} \ d^s < d^{s(t)}. \quad \square$$

2.7. **A version of Ruelle's theorem.** The main result of this section can be extended to general transfer operators with Sobolev weight functions. Repeating the argument of Lemma 2.3 and Lemma 2.5 with obvious simple changes, we obtain the following statement.

Proposition. *Let F be a rational function, and let g be a non-negative continuous function on the Riemann sphere such that g vanishes at the critical points of F and belongs to some Sobolev space $W_{1,q}(\hat{\mathbb{C}})$ with $q > 2$. Then for all numbers $p > 2$ sufficiently close to 2, the condition*

$$P(p' \log \, [g \ |F'|^{\frac{2}{p}-1}]) < p' P(\log g), \qquad \left(p' := \frac{p}{p-1}\right), \tag{2.6}$$

implies the quasicompactness of the transfer operator L_g in $W_{1,p}(\hat{\mathbb{C}})$.

Corollary. *Let F and g be as above, and let λ denote the spectral radius of L_g in $C(J_F)$. Suppose also that g satisfies Ruelle's condition* (1.4)*:*

$$\exists n : \quad \lambda^n > \sup_{J_F} g_n .$$

Then for all numbers $p > 2$ sufficiently close to 2, the operator L_g acts in $W_{1,p}(\hat{\mathbb{C}})$ and is quasicompact.

Proof: Since g vanishes at the critical points of F, we can represent it as follows:

$$g \;=\; h\,|F'|^{\tau},$$

where τ is some positive number and $h \in W_{1,q}$ for some $q > 2$. By the argument of Lemma 2.3, the transfer operator acts in $W_{1,p}$ provided that

$$2 < p < q \quad \text{and} \quad p < \frac{4}{2-\tau}$$

(we can assume $\tau < 2$). It remains to show that (1.4) implies (2.6) for all p close to 2.

If the condition (1.4) is true, then there is $\lambda_1 < \lambda$ such that

$$\|g_n\|_\infty \;\lesssim\; \lambda_1^n .$$

It follows that for all sufficiently small $\varepsilon > 0$, we have

$$\|\, g_n\, |F_n'|^{-\varepsilon} \|_\infty \;=\; \|\, h_n^{\frac{\varepsilon}{\tau}}\, g_n^{\frac{\tau-\varepsilon}{\tau}} \|_\infty \;\lesssim\; \lambda_2^n$$

for some $\lambda_2 < \lambda$. Given p close to 2, we set $\varepsilon = p - 2$. Then we have

$$\left\| z \;\mapsto \sum_{y \in F^{-n}(z)} g_n(y)^{p'}\, |F_n'(y)|^{(-1+2/p)p'} \right\|_\infty =$$

$$\left\| z \;\mapsto \sum_{y \in F^{-n}(z)} \left[g_n\, |F_n'(y)|^{-\varepsilon} \right]^{p'-1} g_n \right\|_\infty \lesssim$$

$$\lesssim\; \lambda_2^{(p'-1)n} \lambda^{n+o(n)} \;\lesssim\; \lambda_3^{p'n}$$

with some $\lambda_3 < \lambda$. This implies (2.6). □

The last statement represents a version of Ruelle's theorem mentioned in Section 1.2. As we noted, the condition (1.6) is weaker than Ruelle's condition (1.4). The latter condition can fail even if (1.6) is valid.

3. Analyticity of the pressure function.

In this section we verify the statements of Theorems A and B for non-exceptional polynomials.

3.1.

Theorem. *If F is not exceptional, then the function $s(t)$ is real analytic for $t < 0$.*

Again, the proof is rather standard. It is contained in the next four lemmas. Fix $t < 0$ and $p > 2$ satisfying the condition of Lemma 2.3. Denote

$$\lambda \equiv \lambda(t) := d^{s(t)}.$$

In other words,

$$\lambda = \rho(L_t, C(\bar{\Omega})) = \rho(L_t, W_{1,p}(\Omega)).$$

We show that $\lambda(t)$ is an isolated, simple eigenvalue of $L_t : W_{1,p}(\Omega) \to W_{1,p}(\Omega)$. Then the theorem follows by the usual application of the analytic perturbation theory. The first lemma is taken from [32]. Lemma 3.3 is a version of the construction of conformal measures due to Patterson [26] and Sullivan [35]. Lemma 3.5 is essentially Lemma 6.1 of [21]. For the convenience of the reader, we outline the proofs.

3.2. **λ is an eigenvalue.**

Lemma. *We have $ker(L_t - \lambda) \neq \emptyset$ in $W_{1,p}(\Omega)$. The corresponding eigenspace contains a non-negative eigenfunction.*

Proof: Since $\rho_{\text{ess}}(L_t, W_{1,p}(\Omega)) < \lambda$, there are only finitely many eigenvalues λ_j satisfying $|\lambda_j| = \lambda$, and the corresponding spectral projections have finite ranks. Denote

$$g_j := P_j\ 1; \quad g_0 := 1 - \sum g_j.$$

Applying L_t^n, we have

$$L_t^n\ g_0\ +\ \sum L_t^n\ g_j\ =\ L_t^n\ 1,$$

and since

$$\|L_t^n\ g_0\|_\infty\ \lesssim\ \|L_t^n\ g_0\|_{1,p}\ =\ o(\|L_t^n\ 1\|_\infty) \quad \text{as} \quad n \to \infty,$$

at least one of g_j's is not zero.

We also have

$$\|L_t^n\ g_j\|_{1,p}\ \asymp\ n^{k_j}\ \lambda^n \quad \text{as} \quad n \to \infty,$$

where $k_j \geq 0$ is the maximal integer number such that

$$\varphi_j := (L - \lambda_j)^{k_j}\ g_j \neq 0,$$

(i.e. k_j is the size of the corresponding Jordan cell). Let $k := \max\{k_j\}$. Then

$$p_n\ :=\ n^{-k}\ (L_t^n\ 1)\ =\ \sum_{j:\ k_j = k} \lambda_j^n\ \varphi_j + o(\lambda^n) \tag{3.1}$$

in $W_{1,p}(\Omega)$ and also in $C(\bar{\Omega})$. Since the functions φ_j are linearly independent, we have

$$\|p_n\|_\infty\ \asymp\ \|p_n\|_{1,p}\ \asymp\ \lambda^n,$$

and we also have $p_n(z_0) \asymp \lambda^n$ for some fixed $z_0 \in \partial\Omega$. Since $p_n \geq 0$, it follows that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} \frac{p_n}{\lambda^n} \right\|_\infty\ \gtrsim\ \frac{1}{N} \sum_{n=1}^{N} \frac{p_n(z_0)}{\lambda^n}\ \asymp\ 1\ .$$

By (3.1), this is possible only if one of the eigenvalues λ_j is positive. □

3.3. **Existence of eigenmeasures.** Let L_t^* denote the adjoint of the operator $L_t : C(\bar\Omega) \to C(\bar\Omega)$. Then L_t^* acts in the space $M(\bar\Omega)$ of finite complex measures according to the following formula:

$$L_t^* : \ \nu \ \mapsto \ \mu \ := \ |F'|^{-t} \, (\nu \circ F).$$

The latter means that

$$|F'|^t \in L^1(\mu),$$

in particular $\mu(\text{Crit } F) \ = \ 0$, and that

$$\nu(FA) \ = \ \int_A |F'|^t \, d\mu$$

for every set A such that F is one-to-one on A and satisfies $A \cap (\text{Crit } F) = \emptyset$. In the special case $\nu = \delta_z$, we have

$$L_t^* \delta_z \ = \ \sum_{y \in F^{-1}(z)} |F'(y)|^{-t} \delta_y. \tag{3.2}$$

Lemma. *There exists a probability measure ν on J_F such that*

$$L_t^* \, \nu \ = \ \lambda(t) \, \nu.$$

Proof: Fix a point $z \in \partial\Omega$ and consider the sequence of positive measures

$$\mu_n \ := \ \lambda^{-n} \, (L_t^*)^n \, \delta_z \ = \ \lambda^{-n} \sum_{y \in F^{-n}(z)} |F'(y)|^{-t} \, \delta_y \ .$$

Clearly, $L_t^* \mu_n \ = \ \lambda \, \mu_{n+1}$, and by the proof of Lemma 3.2, we have

$$\|\mu_n\| \ = \ \lambda^{-n} \, L_t^n 1(z) \ \asymp \ n^k$$

for some integer $k \geq 0$. Next we define

$$\nu_n \ := \ \sum_{j=0}^{n} \mu_n \ ,$$

and take some (weak-$*$) limit point ν of the sequence $\nu_n \, / \, \|\nu_n\|$. Then ν is a probability measure supported on J_F, and since

$$\frac{\|L_t^* \nu_n \ - \ \lambda \nu_n\|}{\|\nu_n\|} \ = \ \frac{\|\lambda(\mu_{n+1} - \mu_0)\|}{\|\nu_n\|} \ \asymp \ \frac{n^k}{n^{k+1}} \ = \ \frac{1}{n} \ \to 0,$$

we have $L_t^* \nu \ = \ \lambda \nu$. □

3.4. **Remark.** The last lemma implies in particular that

$$\rho(L_t, C(J_F)) = \rho(L_t, C(\bar\Omega)).$$

Indeed, λ is an eigenvalue of the adjoint of $L_t : C(J_F) \to C(J_F)$, and therefore $\rho(L_t, C(J_F)) \geq \lambda$. The opposite inequality is obvious:

$$\|L_t^n\|_{C(J_F)} = \|L_t^n \ 1\|_{C(J_F)} \leq \|L_t^n \ 1\|_{C(\bar\Omega)}.$$

3.5. **The support of an eigenmeasure.**

Lemma. *Let ν be a probability measure on J_F satisfying*

$$L_t^* \nu \;=\; \lambda(t)\, \nu.$$

Then either

$$\text{supp } \nu \;=\; J_F,$$

or the set

$$\Sigma := \text{supp}\, \nu$$

is finite and satisfies

$$F^{-1}\Sigma \setminus \text{Crit } F = \Sigma,$$

in particular F is exceptional. In the latter case, we have (see Introduction for notation)

$$\log \lambda(t) \;=\; -t\chi_* = -t\chi_{\max}.$$

Proof: From the equation

$$\lambda\nu = |F'|^{-t} \nu \circ F \tag{3.3}$$

we have

$$F^{-1}\Sigma \setminus \text{Crit } F \subset \Sigma.$$

It follows that if $\#\Sigma = \infty$, then we can find a point $a \in \Sigma$ such that

$$\bigcup_{n \geq 0} F^{-n} a \subset \Sigma,$$

which implies

$$\Sigma = J_F.$$

On the other hand, if $\#\Sigma < \infty$, then by (3.3) we have

$$\begin{aligned}(x \in \Sigma) \;\Rightarrow\; (\nu(x) \neq 0) \;\Rightarrow\; (|F'(x)| \neq 0 \quad \text{and} \quad \nu(Fx) \neq 0)\\ \Rightarrow\; (x \in F^{-1}\Sigma \setminus \text{Crit } F).\end{aligned}$$

The prove the last statement of the lemma, observe that if $b \in \text{Per } F$, then clearly

$$\log \lambda(t) \;\geq\; -t\chi_b.$$

On the other hand, we have

$$\log \lambda(t) \;=\; -t\chi_a$$

for every periodic point $a \in \Sigma$. □

3.6. **Multiplicity of λ.**

Lemma. *Suppose there exists a probability measure ν such that*

$$L^* \nu \;=\; \lambda(t)\, \nu \quad \textit{and} \quad \text{supp } \nu \;=\; J_F.$$

Then $\lambda \equiv \lambda(t)$ is a simple eigenvalue of the operator L_t in $W_{1,p}(\Omega)$:

$$\dim \; \ker \; (L_t - \lambda)^2 \;=\; 1\;.$$

Proof: We will need the following fact: if $f \in W_{1,p}(\Omega)$, then

$$\begin{cases} L_t f = \lambda f \\ f|_{J_F} = 0 \end{cases} \quad \text{implies} \quad f = 0\,. \tag{3.4}$$

Assuming (3.4), we can use the following standard argument to prove the lemma. It is known that the existence of an eigenmeasure with supp $\nu = J_F$ implies dim ker $(L_t - \lambda) = 1$ in $C(J_F)$, see for example Section 3.6 of [21]. By (3.4), the same is true for the space $W_{1,p}(\Omega)$. Suppose now that

$$(L_t - \lambda)^2 h = 0$$

for some $h \in W_{1,p}(\Omega)$. We need to show that $f := (L_t - \lambda)h$ is trivial. By (3.4), it is sufficient to prove $f|_{J_F} = 0$. We have

$$\begin{aligned} (f,\nu) &= (L_t h,\nu) - (\lambda h,\nu) \\ &= (h, L_t^*\nu) - \lambda\,(h,\nu) = 0. \end{aligned}$$

Since dim ker $(L_t - \lambda) = 1$, we can assume (by Lemma 3.2) that $f \geq 0$, and therefore, we have $f = 0$ ν-almost everywhere. The equality $f|_{J_F} = 0$ now follows from the assumption supp $\nu = J_F$.

It remains to prove (3.4). Fix $z \in \Omega$. We have

$$\begin{aligned} |f(z)| &= |\lambda^{-n} L_t^n f(z)| \\ &\leq \lambda^{-n} \sum_{y \in F^{-n}(z)} |F_n'(y)|^{-t}\, |f(y)| \\ &\lesssim \lambda^{-n} \sum_{y \in F^{-n}(z)} |F_n'(y)|^{-t}\, \mathrm{dist}(y, J_F)^{\alpha}, \end{aligned}$$

where $\alpha < -t$ is a fixed positive number such that $W_{1,p}(\Omega) \subset \mathcal{H}_\alpha$, see (2.1). Observe now that

$$\mathrm{dist}(y, J_F) \lesssim |F_n'(y)|^{-1}. \tag{3.5}$$

Indeed, if z is in the basin of attraction to ∞, and $G(\cdot)$ denotes the Green function with pole at infinity, then (3.5) follows from the estimates

$$\begin{aligned} |F_n'(y)|\; |\nabla G(z)| &= d^n\; |\nabla G(y)| \\ &\lesssim \frac{d^n G(y)}{\mathrm{dist}(y, J_F)} \\ &= \frac{G(z)}{\mathrm{dist}(y, J_F)}. \end{aligned} \tag{3.6}$$

On the other hand, if z belongs to some bounded component of $\mathbb{C}\backslash J_F$, then the iterates $\{F^n\}$ are uniformly bounded in the discs $B(y, \mathrm{dist}(y, J_F))$ (the discs lie in the filled-in Julia set), and so (3.5) follows from the Schwarz lemma.

We can now finish the proof of (3.4). From (3.6) and (3.5), we have

$$\begin{aligned} |f(z)| &\lesssim \lambda^{-n} \sum_{y \in F^{-n}(z)} |F_n'(y)|^{-t-\alpha} \\ &\leq d^{-s(t)n}\; d^{s(t+\alpha)n}\; d^{o(n)} \;\to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

because $s(\cdot)$ is strictly decreasing. □

We conclude this section with several remarks concerning some other "hyperbolic" features of *non-exceptional* polynomials.

3.7. **Remarks.** (i) *Perron-Frobenius Theorem.*

The probability eigenmeasure $\nu \equiv \nu_t$ in Lemma 3.3 is unique, and if $f_t \in W_{1,p}(\Omega)$ denotes the non-negative eigenfunction of L_t satisfying

$$\nu_t(f_t) = 1,$$

then the rank one operator

$$\mathcal{P} := (\cdot, \nu_t) f_t$$

is the spectral projection of $L_t : W_{1,p}(\Omega) \to W_{1,p}(\Omega)$ corresponding to the isolated eigenvalue $\lambda \equiv \lambda(t)$. One can show that

$$\rho((I - \mathcal{P})L_t, W_{1,p}(\Omega)) < \lambda, \tag{3.7}$$

which implies that

$$\lambda^{-n} L_t^n \ \to \ \mathcal{P}$$

with exponential rate of convergence in the uniform operator topology.

To prove (3.7), we first observe that the set $\{f_t = 0\}$ is finite. Assume that

$$L_t \hat{f} = \hat{\lambda} \hat{f}$$

for some number $\hat{\lambda}$ of modulus λ and some function $\hat{f} \in W_{1,p}(\Omega)$ with normalization $\nu_t(|\hat{f}|) = 1$. Then we have

$$|\hat{f}| = f_t$$

(use, e.g., the argument of [21], p.142). Define the function $\eta = \eta(z)$ for $z \in J_F \setminus \{f_t = 0\}$ by the equation

$$\hat{f} \ = \ \eta f_t.$$

From the identity

$$(L_t \ f_t)(z) = \frac{\lambda}{\hat{\lambda}\eta(z)} \ (L_t \ \eta f_t)(z),$$

we have

$$\sum_{y \in F^{-1}z} \left(1 - \frac{\lambda \eta(y)}{\hat{\lambda}\eta(z)} \right) f_t(y) |F'(y)|^{-t} = 0,$$

and therefore

$$\frac{\eta(Fy)}{\eta(y)} \ = \ \frac{\lambda}{\hat{\lambda}}$$

except for a finite set of y's. Taking two periodic points with relatively prime periods and with orbits avoiding this finite set, we have $\hat{\lambda} = \lambda$. □

(ii) *Equilibrium states.*

Let μ_t denote the probability measure $f_t \nu_t$. Standard argument shows that μ_t is an ergodic, F-invariant measure. We claim that μ_t is a unique equilibrium state:

$$P(t) = h_t - t\chi_t, \tag{3.8}$$

where we write h_t and χ_t for the entropy and the exponent of μ_t.

The equality (3.8) follows from the Rokhlin-type formula

$$h_t = \int \log J_t \; d\mu_t, \tag{3.9}$$

where

$$J_t := \lambda(t) \frac{f_t \circ F}{f_t} |F'|^t \;\; \in L^1(\mu_t)$$

is the *Jacobian* of μ_t. (We also use the obvious fact that $\log f_t$ is integrable with respect to μ_t.) The formula (3.9) follows from the well-known estimate

$$h_t \geq \int \log J_t \; d\mu_t$$

and from the variational principle.

To prove the uniqueness result, it is sufficient to show that if μ is an equilibrium state, then

$$\mu(\Psi) = \mu_t(\Psi) \qquad \text{for all} \qquad \Psi \in C^\infty.$$

The latter is an immediate consequence (cf. [28]) of the differentiability at 0 of the pressure function

$$p(s) := P(-t \log |F'| + s\Psi), \qquad (s \in \mathbb{R}),$$

see the next remark and also Section 2.7.

(iii) *Derivatives of the pressure function.*

For non-exceptional polynomials, one can establish the same formulas for the derivatives of $P(t)$ as in the hyperbolic case (see [29, 30, 31]). Namely, for the first derivative we have

$$P'(t) = -\chi_t, \qquad (t < 0),$$

and also

$$P'(0-) = -\chi_m,$$

$$P'(-\infty) = \sup_{\mathcal{M}} \chi_\mu = \lim_{n\to\infty} \frac{1}{n} \log \|F_n'\|_\infty.$$

(Recall that m denotes the measure of maximal entropy.) The first statement follows, for example, from the variational principle which also implies the inequality

$$P'(0-) \geq -\chi_m.$$

To prove that

$$P'(0-) \leq -\chi_m,$$

we denote

$$P_\varepsilon(t) = P(-t \log(|F'| + \varepsilon)),$$

and consider the corresponding equilibrium state $\mu_{\varepsilon,t}$:

$$P_\varepsilon(t) = h_{\varepsilon,t} - t \int \log(|F'| + \varepsilon) \; d\mu_{\varepsilon,t}$$

($h_{\varepsilon,t}$ is the entropy of the equilibrium state). It follows that

$$h_{\varepsilon,t} \;\to\; P_\varepsilon(0) = \log \, d \quad \text{as} \quad t \to 0,$$

and therefore

$$\text{weak*-}\lim_{t\to 0} \; \mu_{\varepsilon,t} \;=\; m$$

by the upper semicontinuity of the entropy and the uniqueness of the maximal measure. Since $P_\varepsilon(t) \le P(t)$, we have

$$\begin{aligned} P'(0-) &\ge \limsup_{t\to 0-} \frac{\log d - P_\varepsilon(t)}{-t} \\ &\ge \limsup_{t\to 0-} \frac{h_{\varepsilon,t} - P_\varepsilon(t)}{-t} \\ &= -\liminf_{t\to 0-} \int \log(|F'|+\varepsilon)\ d\mu_{\varepsilon,t} \\ &= -\int \log(|F'|+\varepsilon)\ dm \quad \to \chi_m \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

To state the formula for the second derivative of the pressure function, we denote $A := \log|F'|$ and $S_n := \sum_{j=0}^{n-1} A \circ F^j$. For $t < 0$, consider the *asymptotic variance* σ_t^2 of the process $\{A \circ F^n\}_{n \ge 0}$ in $L^2(\mu_t)$:

$$\begin{aligned} \sigma_t^2 &:= \lim_{n\to\infty} \frac{1}{n} \int [S_n - \mu_t(S_n)]^2\ d\mu_t \\ &= \int A^2\ d\mu_t + 2\sum_{n=1}^{\infty} \int A(A\circ F^n) d\mu_t. \end{aligned}$$

The asymptotic variance is finite because of the exponential decay of the correlations $\int A(A\circ F^n)d\mu_t$ (use the fact that $L_t(Af_t) \in W_{1,p}(\Omega)$ and apply Perron-Frobenius). As in the hyperbolic case, we have

$$P''(t) = \sigma_t^2\ .$$

Indeed, standard computation based on the differentiation of the identity

$$L_\tau f_\tau = \lambda(\tau) f_\tau$$

(with normalization $\nu_t(f_\tau) \equiv 1$ for the eigenfunctions f_τ) shows that

$$P''(t) \ = \ n^{-1}[\mu_t(S_n^2) - \mu_t(S_n)^2] - < n^{-1} S_n \dot{f}_t, \nu_t >,$$

(the dot denotes the derivative with respect to t) and so we need to show that the last term tends to zero as $n \to \infty$. Since

$$< (A \circ F^j)\dot{f}_t, \nu_t > = \ \lambda(t)^{-j} < A(L_t^j)\dot{f}_t, \nu_t >,$$

we have

$$< n^{-1} S_n \dot{f}_t, \nu_t > = < A M_n \dot{f}_t, \nu_t >,$$

where

$$M_n \dot{f}_t \ := \ \frac{1}{n} \sum_{j=0}^{n-1} \frac{L_t^j}{\lambda(t)^j} \dot{f}_t \quad \stackrel{W_{1,p}}{\longrightarrow} \quad < \dot{f}_t, \nu_t > f_t \ = \ 0.$$

(iv) $P(t) \equiv \tilde{P}(t)$ *for non-exceptional maps.*

This follows from the fact that the equilibrium states μ_t are non-atomic. The latter can be proved as follows. The analyticity of the pressure function implies that

$$P(t) > P'(-\infty)\ t, \qquad (\forall t < 0).$$

On the other hand, we have

$$|P'(-\infty)| \ = \lim_{n\to\infty} \ \frac{1}{n}\log\|F_n'\|_\infty.$$

Hence, for every $t<0$, we have

$$\|F_n'\|_\infty^{-t} = o(\lambda(t)^n) \quad \text{as} \quad n\to\infty.$$

Suppose now that $\nu_t(x)\neq 0$. Since ν_t is an eigenmeasure, we have

$$|F_n'|^{-t}(\nu_t\circ F^n) \ = \ \lambda(t)^n\nu_t,$$

and

$$\nu_t(F^n x) \ = \ \frac{\lambda(t)^n}{|F_n'(x)|^{-t}}\ \nu_t(x) \ \to \ \infty.$$

3.8. **Rigidity.** It follows from Remark (iii) that if $P''(t)=0$ for some $t<0$, then $\sigma_t=0$ and therefore the function $\log|F'|$ is homologous to a constant in $L^2(\mu_t)$, i.e. for some $u\in L^2(\mu_t)$ we have

$$\log|F'| = u - u\circ F + \text{const} \tag{3.10}$$

According to Zdunik [38], $\log|F'|$ can be homologous to a constant in $L^2(m)$, where m is the maximal measure, if and only if F is critically finite and the corresponding orbifold is parabolic. One can modify the argument in [38] to extend her result to our equilibrium states μ_t.

Theorem. *Let F be a nonexceptional rational function. Then*

$$P''(t)>0 \quad \textit{for all} \quad t<0.$$

Proof: Suppose $P''(t)=0$ for some $t<0$ and let $\mu=\mu_t$ denote the corresponding equilibrium state. We claim that (3.10) implies

$$F^{-1}(\mathrm{CV})\subset \mathrm{CV}\cup \mathrm{C}, \tag{3.11}$$

where

$$\mathrm{C} := J\cap \mathrm{Crit}\, F \quad \text{and} \quad \mathrm{CV} := \{F^n c:\ n\geq 1,\ c\in \mathrm{C}\}.$$

It then follows that the set CV is finite, in which case the statement is known.

To prove (3.11), we need the following lemma. Let us choose a subset $S\subset J$ with $\mu S>1/2$ such that u is bounded on S.

Lemma. *Let $p\in J\setminus CV$. Then there is a disc B about p and a subset $E\subset B$ of full μ-measure in B such that the following is true:*

for every pair of points $x,y\in E$, there is an integer $n>0$ and a component P of $F^{-n}B$ such that

(i) the map $F^n:\ P\to B$ is univalent, and

(ii) $x,y\in F^n(S\cap W)$.

This lemma immediately implies (3.11). First we observe that u is bounded on $E\cap\frac{1}{2}B$. Indeed, if $x=F^n a$ and $y=F^n b$ for some $a,b\in S\cap W$, then by (3.10) we have

$$u(x)-u(y) = \log\frac{|F_n'(a)|}{|F_n'(b)|} + u(b) - u(a),$$

and the first term to the right is bounded by the distortion theorem. Next we take $x\in \mathrm{CV}$, $y\in F^{-1}x$ and suppose that $y\notin \mathrm{C}\cup\mathrm{CV}$. It follows that u is μ-bounded in some neighborhood of y. Applying (3.10), we see that u is μ-bounded in some neighborhood of x. On the other hand, there is a critical point $c \in \mathrm{C} \setminus \mathrm{CV}$ such that $x = F^k c$ for some $k \geq 1$. Then u is μ-bounded near c, but the equation

$$\log |F_k'| = u \circ F^k - u + \mathrm{const},$$

shows that u cannot be μ-bounded at x. This proves (3.11) and hence the theorem.

We now turn to the proof of the lemma.

Consider the natural extension $(\tilde{J}, \tilde{F}, \tilde{\mu})$ of the dynamical system (J, F, μ). Recall that $\tilde{F}$ is the left shift in the space of sequences

$$\tilde{J} := \{\tilde{x} = (\dots, x_{-1}, x_0, x_1, \dots) \in J^{\mathbb{Z}} : x_{k+1} = F x_k\}.$$

Let $\pi_k : \tilde{J} \to J$ denote the projection onto the k-th coordinate. We will write π for π_0. The ergodic measure $\tilde{\mu}$ is defined as a unique $\tilde{F}$-invariant measure satisfying $\mu = \pi_* \tilde{\mu}$.

For a given disc B and $n > 0$, we denote by U_{-n} the union of the components of $F^{-n} B$ on which F^n is univalent. Consider the set

$$\mathcal{O} := \{\tilde{x} \in \tilde{J} : x_0 \in B, \ x_k \in U_k \quad \text{for all} \quad k < 0\}.$$

We can introduce a direct product structure in $\mathcal{O}$ in the following way. Let Σ be the set of all infinite sequences of the inverse branches participating in the construction of $\mathcal{O}$:

$$\Sigma = \mathcal{O} / \sim,$$

where, by definition, $\tilde{x} \sim \tilde{y}$ if the points x_k and y_k belong to the same component of U_k for all $k < 0$. If $\tau : \mathcal{O} \to \Sigma$ denotes the corresponding projection, then the map

$$\pi \times \tau : \ \mathcal{O} \ \to \ B \times \Sigma$$

is a bijection.

Consider now the restriction of $\tilde{\mu}$ to the set $\mathcal{O}$ as a measure on $B \times \Sigma$. Let ρ denote the projection of this measure to Σ and $\{\mu_\sigma : \ \sigma \in \Sigma\}$ the corresponding family ("canonical system") of conditional measures on B. The proof of the lemma is based on the following two facts:

(*) *if the radius of B is sifficiently small, then $\tilde{\mu}(\mathcal{O}) > 0$;*

(**) *the restriction of μ to B is absolutely continuous with respect to μ_σ for ρ-a.e. σ.*

Assuming these facts, we can now finish the proof of the lemma. Since $\tilde{\mu}(\pi^{-1} S) > \frac{1}{2}$, applying the ergodic theorem we can find a subset $\mathcal{E}$ of $\mathcal{O}$ of full measure, $\tilde{\mu}(\mathcal{O} \setminus \mathcal{E}) = 0$, such that

$$(\tilde{x}, \tilde{y} \in \mathcal{E}) \quad \Rightarrow \quad (\exists k < 0, \ x_k \in S, \ y_k \in S).$$

Denote

$$E_\sigma := \pi(\mathcal{E} \cap \tau^{-1} \sigma).$$

Then we have

$$0 = \tilde{\mu}(\mathcal{O} \setminus \mathcal{E}) = \int \mu_\sigma(B \setminus E_\sigma) \ d\rho(\sigma),$$

and therefore

$$\mu_\sigma(B \setminus E_\sigma) = 0 \qquad \text{for } \rho\text{-a.e. } \sigma.$$

By (**), we have

$$\mu E_\sigma = \mu B \qquad \text{for } \rho\text{-a.e. } \sigma,$$

and so almost every set E_σ satisfies the condition of the lemma.

It remains to verify (*) and (**).

Proof of (*): Recall that $\mu = f\nu$, where $f = f_t$ and $\nu = \nu_t$ are the corresponding eigenfunction and eigenmeasure respectively. Since $p \notin \mathrm{CV}$, we have $f(p) \neq 0$. We will also use the estimate

$$\| F_n' \|_\infty^{-t} \lesssim \lambda_1^n, \qquad \lambda_1 < \lambda := \lambda(t), \tag{3.12}$$

which is true, as was already mentioned, for all non-exceptional maps. For $n > 0$, let C_{-n} be the union of the components P of $F^{-n}B$ such that

$$P \cap \mathrm{C} \neq \emptyset, \quad \text{but} \quad FP \subset U_{1-n}.$$

It is clear that the number of such components P of C_{-n} as well as the degrees of the maps $F^n: P \to B$ are bounded by a constant depending only on the degree of F. Using the fact that ν is an eigenmeasure and that $f(p) \neq 0$, it follows that if the radius of B is small enough, then

$$\mu C_{-n} \lesssim \text{const}\ \frac{\|f\|_\infty \ \| F_n' \|_\infty^{-t}\ \mu B}{f(p)\ \lambda^n},$$

with a constant depending only on the degree of F. For an arbitrary N, we can take B so small that

$$C_{-1}, \dots, C_{-N} = \emptyset,$$

and by (3.12), we can choose N such that

$$\sum_{n>0} \mu C_{-n} = \sum_{n>N} \mu C_{-n} < \mu B.$$

Since

$$\pi^{-1}B \setminus \mathcal{O} \subset \bigcup_{n>0} \{\tilde{x} \in \pi^{-1}B :\ x_{-n} \in C_{-n}\},$$

we have

$$\tilde{\mu}\mathcal{O} \geq \mu B - \sum \mu C_{-n} > 0.$$

Proof of (**): Fix $\eta \in (0, 1)$, and let B' denote the disc ηB. We will show that if

$$\tilde{\mu}(\mathcal{O} \cap \pi^{-1}B') > 0 \tag{3.13}$$

then

$$(\ \mu e > 0, \quad e \subset B'\ ) \quad \Rightarrow \quad (\mu_\sigma e > 0 \quad \text{for} \quad \rho\text{-a.e.} \quad \sigma).$$

By (*), the inequality (3.13) holds for all η close to 1, and therefore (**) follows.

We will use the symbols P_k, $k > 0$, to denote any component of U_{-k}. The statement follows from the estimate

$$\tilde{\mu}\ [\mathcal{O} \cap \pi_0^{-1}e \cap \pi_{-k}^{-1}P_k] \ \geq\ \text{const}\ \tilde{\mu}\ [\mathcal{O} \cap \pi_0^{-1}B' \cap \pi_{-k}^{-1}P_k], \tag{3.14}$$

with a constant independent of k and P_k. Since

$$\bigcup_{(P_n)} \pi_{-n}^{-1} P_n \searrow \mathcal{O} \qquad \text{as} \quad n \to \infty,$$

we have

$$\begin{aligned} \tilde{\mu}\,[\mathcal{O} \cap \pi_0^{-1} e \cap \pi_{-k}^{-1} P_k] &= \lim_{n \to \infty} \sum \tilde{\mu}\,[\mathcal{O} \cap \pi_{-n}^{-1} P_n] \\ &= \lim_{n \to \infty} \sum \mu(P_n \cap F^{-n} e), \end{aligned} \tag{3.15}$$

where the sums are taken over all components P_n such that $F^{n-k} P_n = P_k$. We can represent the right hand side of (3.14) in a similar way, and so to prove (3.14) we only need to compare the μ-measures of the sets $P_n \cap F^{-n} e$ and $P_n \cap F^{-n} B'$.

Assume first that the eigenfunction f does not vanish on J. Then it is enough to notice that the ν-measures of the above sets are comparable. The latter is a consequence of the distortion theorem and of the fact that ν is an eigenmeasure:

$$\frac{\nu(P_n \cap F^{-n} e)}{\nu(P_n \cap F^{-n} B')} = \frac{\lambda^{-n} \int_e |F_n'|^t \; d(\nu \circ F^n)}{\lambda^{-n} \int_{B'} |F_n'|^t \; d(\nu \circ F^n)}.$$

The eigenfunction f may have zeros in general. Let Z denote the set $\{f = 0\}$. Since F in non-exceptional, there is an integer $m > 0$ such that

$$\delta := \operatorname{dist}(Z, F^{-m} Z) > 0.$$

We can also assume that the disc B is so small that the diameters of all sets $P_n \cap F^{-n} B'$ are $\ll \delta$. Returning to the computation (3.15), we modify some of the terms $\mu(P_n \cap F^{-n} e)$ as follows. If the set $P_n \cap F^{-n} e$ contains a point at which f is very small, then we replace the coresponding term with the sum

$$\sum \mu(P_{n+m} \cap F^{-n-m} e)$$

taken over all components P_{n+m} such that $F^m P_{n+m} = P_n$. In the new expression, the eigenfunction f is bounded away from zero by a constant independent of n, and so the previous argument applies. □

4. Hidden Spectrum

In this section we study the phase transition case, and complete the proof of Theorems A and B.

Let F be an exceptional polynomial. We assume that F not conjugate to a Chebychev's polynomial. From the discussion in Section 1.3, it follows that there exists a fixed point $a \in J_F$, $F(a) = a$, such that

$$F^{-1} a \setminus \{a\} \subset \operatorname{Crit} F.$$

Consider the function

$$H(z) := |z - a|.$$

We have

$$\frac{H \circ F}{H}(z) = \prod_{c \in \operatorname{Crit} F \cap F^{-1} a} |z - c|^{k(c)+1},$$

where $k(c)$ denotes the multiplicity of a critical point c. We also define the number $\tilde{\kappa} > 0$ from the equation

$$\frac{\tilde{\kappa}}{1-\tilde{\kappa}} \;=\; \min\{k(c):\; c \in F^{-1}a \cap \mathrm{Crit}\; F\}.$$

4.1. **The functions $s_\kappa(t)$.** The idea is to replace the weights $|F'|^{-t}$ in the transfer operators (1.1) with "homologous" weights of the form

$$G_{\kappa,t} := |F'|^{-t}\left(\frac{H\circ F}{H}\right)^{\kappa t}.$$

If $0 \le \kappa \le \tilde{\kappa}$, then the weights $G_{\kappa,t}$ are continuous in $\bar{\Omega}$ and the corresponding transfer operators

$$L_{\kappa,t} f(z) := \sum_{y\in F^{-1}(z)} G_{\kappa,t}(y) f(y)$$

are bounded in $C(\bar{\Omega})$. The special property of the case $\kappa = \tilde{\kappa}$ is that every point in Ω has at least one preimage that is not a zero of $G_{\tilde{\kappa},t}$. This means that we are no longer in the "exceptional" situation – we have

$$L^*_{\tilde{\kappa},t}\;\nu \;\neq\; 0 \tag{4.1}$$

for every probability measure ν on J_F. Unfortunately, the operators $L_{\tilde{\kappa},t}$ are not bounded in any space $W_{1,p}(\Omega)$, and to apply the technique of Sections 2 and 3 we have to use $L_{\kappa,t}$ with $\kappa < \tilde{\kappa}$. (The operators with $\kappa < \tilde{\kappa}$ do not satisfy (4.1) but they are bounded in appropriate Sobolev spaces.)

Let $\lambda_\kappa(t)$ denote the spectral radius of $L_{\kappa,t}$ in $C(\bar{\Omega})$. Define

$$s_\kappa(t) := \log_d \lambda_\kappa(t).$$

We will need the following properties of the functions $s_\kappa(t)$.

(i) *If $t < 0$ and $0 \le \kappa \le \kappa' \le \tilde{\kappa}$, then $s_{\kappa'}(t) \le s_\kappa(t)$.*

Proof: Denote

$$h(z) = |z-a|^{-t(\kappa'-\kappa)}$$

and observe that

$$L^n_{\kappa',t}\;1 \;=\; \frac{1}{h}\;L^n_{\kappa,t}\;h.$$

Let z_n be the points in $\partial\Omega$ such that

$$\|L^n_{\kappa',t}\|_\infty \;=\; L^n_{\kappa',t}\;1(z_n).$$

The existence of such points follows from the subharmonicity of the function

$$z \mapsto L^n_{\kappa',t}\;1(z).$$

Then we have

$$\|L^n_{\kappa',t}\|_\infty \;\asymp\; L^n_{\kappa,t}\;h(z_n) \;\le\; \|L^n_{\kappa,t}\;h\|_\infty \;\lesssim\; \|L^n_{\kappa,t}\|_\infty,$$

which implies the statement. □

(ii) *If there is a probability measure ν satisfying*

$$L^*_{\kappa,t}\;\nu = \lambda_\kappa(t)\;\nu$$

and if $\nu \neq \delta_a$, then

$$s_{\kappa'}(t) = s_\kappa(t) \quad \text{for all} \quad \kappa' > \kappa.$$

Proof: We have

$$\begin{aligned} \|L^n_{\kappa',t}\|_\infty &\gtrsim \|h\ L^n_{\kappa',t}\ 1\|_\infty \ = \ \|L^n_{\kappa,t}\ h\|_\infty \\ &\gtrsim < L^n_{\kappa,t}\ h, \nu > = \ \lambda^n_\kappa(t) < h, \nu > \\ &\asymp \ \lambda^n_\kappa(t), \end{aligned}$$

which implies

$$s_{\kappa'}(t) \ \geq \ s_\kappa(t).$$

□

(iii) *For every $\kappa \in [0, \tilde\kappa]$, the function $s_\kappa(\cdot)$ is strictly decreasing.*

Proof: It is clear that $\nu_t \neq \delta_a$ if t is sufficiently close to 0. By the previous statement, we have $s_\kappa(t) = s(t)$ for such t's, and therefore the function $s_\kappa(t)$ is strictly decreasing in a neighborhood of 0. It remains to note that s_κ is convex (use Hölder's inequality and the definition of s_κ). □

4.2.

Lemma. $\tilde s(t) > -t(1-\tilde\kappa)\ \log_d |F'(a)|$.

Proof: Denote $M := F'(a)$. The statement is obvious if a is a neutral fixed point, so we assume that a is repelling: $|M| > 1$. For simplicity, we write G and L instead of $G_{\tilde\kappa,t}$ and $L_{\tilde\kappa,t}$ respectively. Observe that

$$G(a) = |M|^{-t(1-\tilde\kappa)}.$$

By (4.1), we can consider the operator

$$\nu \ \mapsto \ \|L^*\nu\|^{-1}\ L^*\nu$$

on the set of probability measures on J_F. By Schauder's theorem, this operator has a fixed point ν, and we have

$$L^*\nu \ = \ \lambda\,\nu \tag{4.2}$$

for some $\lambda > 0$. It is clear that $\log_d\ \lambda \leq \tilde s(t)$, and it remains to show that

$$G(a) < \lambda. \tag{4.3}$$

Since a is a repelling point of F, there is a conformal map φ from the unit disc onto some neighborhood of a such that

$$\varphi(Mz) = F(\varphi(z)), \qquad (|z| < |M|^{-1}).$$

If $|z| < |M|^{-(1+n)}$, then we have

$$|F_n'(\varphi(z))| = |M|^n \frac{|\varphi'(M^n z)|}{|\varphi'(z)|} \asymp |M|^n,$$

and

$$\begin{aligned} G_n(\varphi(z)) \ &= \ |F_n'|^{-t} \left(\frac{|\varphi(M^n z) - a|}{|\varphi(z) - a|}\right)^{\tilde\kappa t} \\ &\asymp \ |M|^{-tn}\ (|M|^n)^{\tilde\kappa t} \ = \ G(a)^n. \end{aligned}$$

To prove (4.3), we consider the sequence of pairwise disjoint domains

$$U_n \ := \ \varphi\,(|M|^{-(2+n)} < |z| < |M|^{-(1+n)}), \qquad (n \geq 0).$$

By construction, F^n is injective on U_n, $F^n(U_n) \ = \ U_0$, and

$$G_n(z) \asymp G(a)^n \quad \text{for} \quad z \in U_n.$$

Then by (4.2), we have

$$\begin{aligned}\nu(U_n) \ &= \ \lambda^{-n} \int_U G_n(z)\, d\nu(z) \\ &\asymp \ \lambda^{-n}\, G(a)^n \ \nu(U).\end{aligned}$$

It is easy to see that supp $\nu \ = \ J_F$. (This follows from (4.1), see the proof of Lemma 3.5.) Hence $\nu(U) > 0$, and since the domains U_n are disjoint, we have

$$\sum_{n\geq 0} \left(\frac{G(a)}{\lambda}\right)^n \ \lesssim \ \sum_{n\geq 0} \nu(U_n) \ < \ 1,$$

which implies (4.3). □

4.3. **The operators $L_{\kappa,t}$ with $\kappa < \tilde\kappa$.** The argument of Lemma 2.3 shows that if $t < 0$ and $0 \leq \kappa < \tilde\kappa$, then $L_{\kappa,t}$ is bounded in $W_{1,p}(\Omega)$ with $p > 2$ sufficiently close to 2. We can now apply the methods of Sections 2 and 3 to establish the following result. The condition (4.4) below simply means that a measure ν satisfying

$$L^*_{\kappa,t}\nu \ = \ \lambda_\kappa(t)\ \nu$$

cannot be equal to δ_a, and therefore

$$\text{supp}\ \nu = J_F$$

by the proof of Lemma 3.5. Indeed, we have

$$L^*_{\kappa,t}\delta_a = G_{\kappa,t}(a)\ \delta_a,$$

and if we assume (4.4), then

$$G_{\kappa,t}(a) = |F'(a)|^{-t(1-\kappa)} \ < \ \lambda_\kappa(t).$$

Lemma. *Let $0 \leq \kappa < \tilde\kappa$, and $t < 0$. Suppose that*

$$s_\kappa(t) > -t(1-\kappa)\ \log_d |F'(a)|. \tag{4.4}$$

Then the function $s_\kappa(\cdot)$ is real analytic at t, and there is a non-atomic equilibrium state $\mu_{\kappa,t}$ for the function $\log G_{\kappa,t}$.

Proof: There are only minor changes in the reasoning of the previous sections. We again write G and L for $G_{\kappa,t}$ and $L_{\kappa,t}$.

(i) We first establish a two-norm inequality similar to (2.4). Choose $p > 2$ such that L acts in $W_{1,p}(\Omega)$. We claim that for some $\varepsilon > 0$,

$$\|L^n f\|_{1,p} \leq d^{n(s_\kappa(t)-\varepsilon)+o(n)}\|f\|_{1,p} + C_n\|f\|_\infty\,, \quad (f \in W_{1,p}(\Omega)). \tag{4.5}$$

To prove (4.5), we repeat the computation of Lemma 2.5 to obtain

$$\int_\Omega |\nabla(L^n f)|^p \;\lesssim\; \|\hat{L}^n\|^{\frac{p}{p'}} \|\nabla f\|_{1,p}^p \;+\; C_n \|f\|_\infty^p, \tag{4.6}$$

where $\hat{L}$ denotes the transfer operator

$$\hat{L}f(z) = \sum_{y \in F^{-1}z} f(y)\ \hat{G}(y)$$

with the weight function

$$\hat{G} \;:=\; G^{p'}\ |F'|^{(\frac{2}{p}-1)p'} \;\equiv\; G_{\hat{\kappa},\hat{t}},$$

p' is the conjugate exponent, and

$$\hat{t} := p'\left(t + 1 - \frac{2}{p}\right), \qquad \hat{\kappa} := \frac{\kappa t}{t + 1 - \frac{2}{p}}.$$

Since $\hat{\kappa} > \kappa$ and $\hat{t} > p't$, the properties (i) and (iii) of Section 4.1 imply that

$$\frac{1}{p'}\ s_{\hat{\kappa}}(\hat{t}) \;<\; \frac{1}{p'}\ s_\kappa(p't) \;\le\; s_\kappa(t),$$

and therefore

$$\|\hat{L}^n\|^{\frac{1}{p'}} \;=\; d^{n(s_\kappa(t)-\varepsilon)+o(n)}.$$

Together with (4.6), the latter implies (4.5).

(ii) The quasicompactness of L,

$$\rho_{\text{ess}}(L, W_{1,p}(\Omega)) \;<\; \rho(L, W_{1,p}(\Omega)) = \rho(L, C(\bar{\Omega})) \equiv \lambda_\kappa(t),$$

is a consequence of the two-norm inequality (4.5). It also follows that $\lambda_\kappa(t)$ is an eigenvalue of $L : W_{1,p}(\Omega) \to W_{1,p}(\Omega)$ and that there is a probability measure $\nu_{\kappa,t}$ satisfying

$$L^*\nu_{\kappa,t} \;=\; \lambda_\kappa(t)\ \nu_{\kappa,t}.$$

The proofs are identical to those in Sections 2 and 3. As we mentioned, from (4.4) we have

$$\text{supp}\ \nu_{\kappa,t} = J_F. \tag{4.7}$$

This in turn implies that $\lambda_\kappa(t)$ is a simple isolated eigenvalue of $L : W_{1,p}(\Omega) \to W_{1,p}(\Omega)$, and so the spectrum $s_\kappa(\cdot)$ is analytic at t. The proof is exactly the same as in Lemma 3.6 except that the fact

$$(\ f\ \in\ W_{1,p}(\Omega), \quad Lf = \lambda_\kappa(t)\ f, \quad f|_{J_F} \equiv 0\ ) \quad \Rightarrow \quad (f \equiv 0) \tag{4.8}$$

requires a slightly different argument. Fix $z \in \Omega \setminus J_F$. Then we have

$$\begin{aligned} |f(z)| = &\ |\lambda_\kappa(t)^{-n}\, L^n f(z)| \\ \lesssim &\ \lambda_\kappa^{-n}(t) \sum_{y \in F^{-n}(z)} |G_n(y)|\ \text{dist}(y, J_F)^\beta, \end{aligned}$$

for some positive number $\beta < -t$. Using the inequality (3.5), we have

$$\begin{aligned} |f(z)| &\lesssim \lambda_\kappa^{-n}(t) \sum_{y \in F^{-n}(z)} G_n(y)\ |F_n'(y)|^{-\beta} \\ &= \lambda_\kappa^{-n}(t)\ H(z)^{t\kappa} \sum_{y \in F^{-n}(z)} |F_n'(y)|^{-t-\beta}\ H(y)^{-t\kappa} \\ &= e^{o(n)}\ d^{-s_\kappa(t)n}\ d^{s_{\hat\kappa}(\hat t)n}, \end{aligned}$$

with

$$\hat t := t + \beta > t, \quad \text{and} \quad \hat\kappa = \frac{t}{t+\beta} > \kappa.$$

By (i) and (iii) of Section 4.1, we have

$$s_{\hat\kappa}(\hat t) < s_\kappa(t),$$

which completes the proof of (4.8).

(iii) The construction of an equilibrium state μ and the proof that μ has no atoms is the same as in Section 3.7. □

4.4. **Corollary.** $\tilde P(t) = \tilde s(t)\ \log d$.

Proof: Fix $t < 0$. By property (i) of Section 4.1 and by Lemma 4.2, we have

$$s_\kappa(t) \geq \tilde s(t) > -t(1-\tilde\kappa) \log_d |F'(a)|,$$

and therefore

$$s_\kappa(t) > -t(1-\kappa) \log_d |F'(a)|$$

for some parameter $\kappa \in (0, \tilde\kappa)$ which we now consider fixed. As we mentioned, the last inequality implies that there exists an eigenmeasure $\nu_{\kappa,t}$ satisfying $\operatorname{supp} \nu_{\kappa,t} = J_F$. By property (ii), it follows that

$$\tilde s(t) = s_\kappa(t).$$

Applying the variational principle (see Section 1.2), we have

$$s_\kappa(t)\ \log d = P(\log G_{\kappa,t}).$$

We also have the equality

$$\tilde P(t) = P(\log G_{\kappa,t})$$

which follows from the existence of a non-atomic equilibrium state for the function $\log G_{\kappa,t}$ and from the fact that if μ is a probability measure on J_F such that $\mu(a) = 0$, then

$$\mu(\log G_{\kappa,t}) = -t\chi_\mu. \tag{4.9}$$

To prove (4.9), we observe that if

$$\log \frac{H \circ F}{H} \notin L^1(\mu),$$

then both sides in (4.9) are $-\infty$, otherwise we have

$$\mu\left(\log \frac{H \circ F}{H}\right) = 0.$$

Indeed, for $\varepsilon \in (0,1)$ denote $H_\varepsilon := H + \varepsilon$. Then

$$\left|\log \frac{H_\varepsilon \circ F}{H_\varepsilon}\right| \leq \left|\log \frac{H \circ F}{H}\right| + \text{const}$$

on J_F, and

$$\log\frac{H_\varepsilon\circ F}{H_\varepsilon}\ \overset{\mu\text{-a.e.}}{\longrightarrow}\ \log\frac{H\circ F}{H}\qquad\text{as}\quad \varepsilon\to 0.$$

□

4.5. **Proof of Theorems A and B.** If F is not exceptional, then $P_F(t)$ is real analytic on the negative axis, and therefore $P_F(t)>-\chi_{\max}$ for all $t<0$. The equality $P_F=\tilde{P}_F$ was explained in Section 3.7.

Suppose now that F is an exceptional map. Clearly, we always have

$$P_F(t)\ \geq\ \max\{\tilde{P_F}(t),-\chi_{\max}t\}.$$

If $P_F(t)>-\chi_* t$ for some $t<0$, then we have $P_F(t)=\tilde{P}_F(t)$ by the property (ii) and Lemma 3.5. This completes the proof of Theorem A.

A phase transition occurs if and only if

$$\chi_*>\tilde{P}'_F(-\infty).$$

On the other hand, it is clear that

$$\tilde{P}'_F(-\infty)=\sup\{\chi_\mu:\ \mu\in\mathcal{M},\ \mu(\Sigma_F)=0\},$$

and Theorem B follows.

4.6. **Remark.** One can extend all results of Sections 3.7 and 3.8 to exceptional polynomials. In particular, the argument of Section 3.8 proves
Theorem C: $\tilde{P}''(t)>0$ for all $t<0$ unless F is critically finite with parabolic orbifold. In the next section we will also use the following formula involving $\tilde{P}'(t)$.

For $t<0$, let κ be a number satisfying the conditions of Lemma 4.3, and let $\mu\equiv\mu_{\kappa,t}$ be the corresponding equilibrium state. Then applying (4.9), we have

$$\tilde{P}'(t)=-\chi_\mu.$$

Since

$$\tilde{P}(t)=h_\mu-t\chi_\mu,$$

we get

$$\dim\mu=\frac{h_\mu}{\chi_\mu}=t-\frac{\tilde{P}(t)}{\tilde{P}'(t)}. \tag{4.10}$$

(The first equality in (4.10) follows from Mañé's formula [23].)

5. Dimension spectrum

In this section we study the dimension properties of the maximal measure m and prove Theorem D.

5.1. **Definitions and results.** We define the *box-counting dimension spectrum* $f(\alpha)$ of m as follows:

$$f(\alpha) := \lim_{\eta\to 0} \limsup_{\delta\to 0} \frac{\log N(\delta;\alpha,\eta)}{|\log\delta|},$$

where $N(\delta;\alpha,\eta)$ is the maximal number of disjoint discs B of radius δ centered at J_F and satisfying

$$\delta^{\alpha+\eta} \le mB \le \delta^{\alpha-\eta}.$$

The *Hausdorff dimension spectrum* $\tilde f(\alpha)$ is defined be the equation

$$\tilde f(\alpha) := \dim\,\{z : \alpha(z) \text{ exists and } = \alpha\},$$

where $\alpha(z)$ is the pointwise dimension of m at z, and dim denotes Hausdorff dimension if the set is uncountable and $-\infty$ otherwise. Recall the statement of Theorem D. Let α_0 denote the Hausdorff dimension of the maximal measure. By (iii) of Section 3.7, we have $\alpha_0 = |s'(0-)|^{-1}$.

Claim. *(i) The function $s(t)$ on $\{t \le 0\}$, and the function $f(\alpha)$ on $\{\alpha \le \alpha_0\}$ form a Legendre pair:*

$$\begin{aligned} s(t) &= \sup_{\alpha\le\alpha_0} \frac{f(\alpha)-t}{\alpha}, \qquad (t \le 0),\\ f(\alpha) &= \inf_{t\le 0}\,[t + \alpha s(t)], \qquad (\alpha \le \alpha_0). \end{aligned}$$

(ii) The functions $\tilde s(t),\ t \le 0$, and $\tilde f(\alpha),\ \alpha \le \alpha_0$ form a Legendre pair.

Using Theorem D, we can restate our results on the pressure function in terms of the spectra $f(\alpha)$ and $\tilde f(\alpha)$. Let us assume that F is not critically finite with parabolic orbifold. Denote

$$\alpha_{\min} := \frac{1}{|s'(-\infty)|}.$$

If $s(t)$ has a phase transition point, then we also define the parameters

$$\tilde\alpha_{\min} := \frac{1}{|\tilde s'(-\infty)|}$$

and

$$\alpha_c := \frac{1}{|s'(t_c+)|} = \frac{1}{|\tilde s'(t_c)|}.$$

We always have

$$0 < \alpha_{\min} < \alpha_0,$$

and in the phase transition case we have

$$0 < \alpha_{\min} < \tilde\alpha_{\min} < \alpha_c < \alpha_0.$$

Finally, note that $f(\alpha_0) = \tilde f(\alpha_0) = \alpha_0$ because $\alpha_0 = \dim m$.

Corollary 1. *If F is not critically finite with parabolic orbifold, then $\tilde f(\alpha)$ is a real analytic, strictly increasing and strictly convex ($\tilde f'' > 0$) function on the interval $(\tilde\alpha_{\min}, \alpha_0)$, and $\tilde f(\alpha) \equiv -\infty$ for $\alpha < \tilde\alpha_{\min}$.*

Proof: Define

$$\alpha(t) := |\tilde s'(t)|^{-1}.$$

Since $\tilde{s}'' > 0$, we have

$$\alpha'(t) = \frac{\tilde{s}''(t)}{(\tilde{s}'(t))^2} > 0,$$

and so $\alpha(t)$ is strictly increasing on the interval $(-\infty, 0)$, and the inverse function $t(\alpha)$ is real analytic on $(\tilde{\alpha}_{\min}, \alpha_0)$. It follows that for $\alpha \in (\tilde{\alpha}_{\min}, \alpha_0)$, the function

$$\begin{aligned} \tilde{f}(\alpha) &= \inf_{t \le 0} [t + \alpha \tilde{s}(t)] \\ &= t(\alpha) + \alpha \tilde{s}(t(\alpha)) \end{aligned}$$

has the stated properties. It is also clear that $\tilde{f}(\alpha) \equiv -\infty$ if $\alpha < \tilde{\alpha}_{\min}$. □

Corollary 2. *If F is not exceptional (more generally, if there is no phase transition), then*

$$f \equiv \tilde{f}.$$

In the phase transition case, $f(\alpha)$ is C^1 but not C^2 on $(\alpha_{\min}, \alpha_0)$. More precisely,

$$f(\alpha) = \begin{cases} \tilde{f}(\alpha), & \alpha_c \le \alpha \le \alpha_0, \\ \text{linear}, & \alpha_{\min} \le \alpha \le \alpha_c, \\ 0, & \alpha = \alpha_{\min}, \\ -\infty, & \alpha < \alpha_{\min}. \end{cases}$$

Proof: Reasoning as above, we have

$$f(\alpha) = t + \alpha s(t), \qquad (\alpha_c \le \alpha \le \alpha_0),$$

where α and t are related by the equation $\alpha s'(t) = -1$. We also have

$$f(\alpha) = t_c \left(1 - \frac{\alpha}{\alpha_{\min}}\right) \quad \text{on} \quad [\alpha_{\min}, \alpha_c].$$

It follows that f' is continuous at α_c. Indeed,

$$\begin{aligned} f'(\alpha_c+) &= \frac{1}{\alpha_c}[f(\alpha_c) - t_c] \\ &= \frac{1}{\alpha_c}\left[t_c\left(1 - \frac{\alpha_c}{\alpha_{\min}}\right) - t_c\right] \\ &= -\frac{t_c}{\alpha_{\min}} = f'(\alpha_c-). \end{aligned}$$

The rest of the proof is obvious. □

We will prove the theorem only for *polynomials* with *connected* Julia sets. The proof is considerably shorter in this special case because we can express the spectra $s(t)$ and $\tilde{s}(t)$ in terms of the Riemann map

$$\varphi : \Delta := \{|z| > 1\} \to A(\infty), \qquad (\varphi(\infty) = \infty),$$

where $A(\infty)$ is the basin of attraction to infinity, and apply some general facts of the conformal mapping theory. (For arbitrary rational maps, one should replace certain parts of the argument with corresponding dynamical considerations.) Recall that for connected polynomial Julia sets, m is the image of the normalized Lebesgue measure under the boundary correspondence. In what follows, we assume that the polynomial F is exceptional (but not Chebychev's) with $\Sigma_F = \{a\}$.

5.2.

Lemma. *For each $t<0$, we have*

$$d^{ns(t)} \asymp d^{n(1-t)} \int_{|z|=1+d^{-n}} |\varphi'|^t, \tag{5.1}$$

$$d^{n\tilde{s}(t)} \asymp d^{n(1-t)} \int_{|z|=1+d^{-n}} |\varphi-a|^{-\tilde{\kappa}t}\ |\varphi'|^t. \tag{5.2}$$

Proof: Fix some point in $A(\infty)$ and consider the preimages $\{y\}$ under F^n.
The Riemann map φ conjugates F with the dynamics $T:\ z\mapsto z^d$ on Δ.
Differentiating the identity $F^n\circ\varphi = \varphi\circ T^n$, we get

$$|F_n'(y)| \asymp d^n |\varphi'(\varphi^{-1}y)|^{-1}.$$

The points $\{\varphi^{-1}y\}$ are equidistributed on a circle of radius r_n satisfying

$$r_n - 1 \asymp d^{-n}.$$

Applying the distortion theorem, we have

$$\|L_t^n\|_\infty \asymp d^{n(1-t)} \int_{|z|=r_n} |\varphi'|^t,$$

and

$$\begin{aligned}\|L_{\tilde{\kappa},t}^n\|_\infty &\asymp \sum_y |F_n'(y)|^{-t}|y-a|^{-\tilde{\kappa}t}\\ &\asymp d^{n(1-t)} \int_{|z|=r_n} |\varphi-a|^{-\tilde{\kappa}t}\ |\varphi'|^t.\end{aligned}$$

□

5.3. **Proof of (i).** The key observation is that $s(t)$ coincides with the *packing spectrum* of the maximal measure m:

$$\pi(t) = \limsup_{\varepsilon\to 0} \frac{\log L(\varepsilon;t)}{|\log\varepsilon|},$$

where

$$L(\varepsilon;t) := \sup_{\mathcal{B}} \sum_{B\in\mathcal{B}} \operatorname{diam}(B)^t$$

the supremum being taken over all collections $\mathcal{B}$ of disjoint discs B satisfying $mB=\varepsilon$. It is a general fact (see [20]) that the harmonic measure packing spectrum of an arbitrary simply connected domain is related to the integral means spectrum

$$\beta(t) := \limsup_{r\to 1} \frac{\int_{|z|=r} |\varphi'(z)|^t\ |dz|}{|\log(r-1)|}$$

of the corresponding conformal map by the equation

$$\pi(t) = \beta(t) + 1 - t.$$

Thus for polynomials with connected Julia set, the equality $s(t)=\pi(t)$ follows from (5.1). The packing spectrum and the box-counting dimension spectrum of an arbitrary measure satisfy the Legendre-type relation

$$s(t) = \sup_{\alpha\le\dim m} \frac{f(\alpha)-t}{\alpha}, \quad (t\le 0),$$

and so we obtain the first formula in (i).

Applying the inverse Legendre transform , we get

$$\text{co } f(\alpha) = \inf_{t<0} [t + \alpha s(t)],$$

where co f denotes the *convex envelope* of f. Since $s(t)$ is differentiable and strictly convex on $(t_c, 0)$, we have

$$f(\alpha) \equiv \text{co } f(\alpha) \qquad \text{on} \quad (\alpha_c, \alpha_0),$$

and to finish the proof, it remains to show that

$$f(\alpha) \geq \text{co } f(\alpha) \equiv t_c \left(1 - \frac{\alpha}{\alpha_{\min}}\right) \qquad \text{on} \quad (\alpha_{\min}, \alpha_c). \tag{5.3}$$

To prove (5.3), we fix $\alpha \in (\alpha_{\min}, \alpha_c)$ and consider a neighborhood U of a such that the dynamics $F|_U : U \to FU$ is conjugate to the map

$$z \mapsto F'(a) : \{|z| < 1\} \to \{|z| < e^{\chi_*}\}.$$

(Recall that $\chi_* = \log|F'(a)|$ and $\alpha_{\min} = \chi_*^{-1} \log d$.) For a small number δ let N be the maximal number of disjoint discs $B \subset U$ of radius δ and harmonic measure $\geq \delta^{\alpha_c}$. We have

$$N \geq \left(\frac{1}{\delta}\right)^{f(\alpha_c)-\varepsilon}$$

with ε arbitrarily small (as $\delta \to 0$). Let k be an integer number such that

$$k \approx \frac{1}{\chi_*} \frac{\alpha_c - \alpha}{\alpha - \alpha_{\min}} \log \frac{1}{\delta}.$$

Applying $(F|_U)^{-k}$ to the discs B, we find N new discs of radius

$$\asymp \delta_{(k)} := e^{-\chi_* n} \delta$$

and harmonic measure

$$\geq d^{-n} \delta^{\alpha_c} = \delta_{(k)}^{\alpha}.$$

It follows that

$$\begin{aligned} f(\alpha) &\geq f(\alpha_c) \limsup_{\delta \to 0} \frac{\log \delta}{\log \delta_{(k)}} \\ &= f(\alpha_c) \frac{\alpha - \alpha_{\min}}{\alpha_c - \alpha_{\min}} \\ &= t_c \left(1 - \frac{\alpha}{\alpha_{\min}}\right). \end{aligned}$$

5.4. **Proof of (ii).** Let us now prove the statement concerning the Hausdorff dimension spectrum. For $\varepsilon > 0$, let U_ε denote the ε-neighborhood of the exceptional point a, m_ε the restriction of the maximal measure m to $J \setminus U_\varepsilon$, and let $\pi_\varepsilon(t)$ and $f_\varepsilon(\alpha)$ be the packing and the box dimension spectra of m_ε. As we mentioned, $\pi_\varepsilon(t)$ is the Legendre transform of $f_\varepsilon(\alpha)$. From (5.2) it is easy to see that

$$\pi_\varepsilon(t) \leq \tilde{s}(t).$$

Applying the inverse transform to this inequality, we have

$$\text{co } f_\varepsilon \ \leq \ \inf_{t \leq 0} \ [t + \alpha\tilde{s}(t)].$$

On the other hand, it is clear that the Hausdorff spectrum $\tilde{f}(\alpha)$ satisfies the inequality

$$\tilde{f}(\alpha) \ \leq \ \sup_{\varepsilon > 0} f_\varepsilon(\alpha),$$

and therefore we have

$$\tilde{f}(\alpha) \ \leq \ \inf_{t \leq 0} \ [t + \alpha\tilde{s}(t)].$$

To finish the proof, we need to verify the opposite inequality.

Fix $\alpha \in (\tilde{\alpha}_{\min}, \alpha_0)$ and define $t = t(\alpha)$ by the equation

$$\alpha\tilde{s}'(t) + 1 = 0.$$

We will show that

$$\dim\{z : \underline{\alpha}(z) \geq \alpha\} \ \geq \ \alpha\tilde{s}(t) + t = \ t - \frac{\tilde{s}(t)}{\tilde{s}'(t)}.$$

Let κ be a number satisfying the conditions of Lemma 4.3, and let $\mu \equiv \mu_{\kappa,t}$ be the corresponding equilibrium state. By (4.10), we have

$$\dim \mu = t - \frac{\tilde{s}(t)}{\tilde{s}'(t)}.$$

On the other hand standard ergodic argument shows that for μ-a.e. z, we have

$$\underline{\alpha}(z) \geq \frac{\log d}{\chi_\mu} = -\frac{1}{\tilde{s}'(t)} = \alpha.$$

This completes the proof of Theorem D.

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