# Recovering modular forms from squares

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The purpose of this appendix[1] is to provide a proof of the fact that a holomorphic newform $f$ of weight $2k$, level $N$ and trivial character, with Hecke eigenvalues $\{a_p \,|\, (p,N)=1\}$, is determined up to a quadratic twist, in fact *on the nose* if $N$ is square-free, by the knowledge of $a_p^2$ for all primes $p$ in a set of sufficiently large density. We will in fact prove a more general statement below, including the case of odd weight and non-trivial character, and also establish a mod $\ell$ analog. We found this result in the summer of 94, and we have since learned that it has also been known to others, including Don Blasius and J.-P. Serre. Also, Siman Wong has recently come up with a different proof in the weight 2 case (with trivial character). So we do not intend any display of great achievement by this write-up, and we give all the details for ease of use by those working in classical modular forms and number theory. We have also found a non-trivial extension of this result (in characteristic zero) to Maass forms using an array of results on automorphic $L-$functions, and this is the subject matter of a paper under preparation. This work was partially supported by an NSF grant. We thank Serre for his helpful comments on an earlier version which led to a finer result.

For every pair of integers $N, k \geqslant 1$, and character $\omega : (\mathbf{Z}/N)^* \to \mathbf{C}^*$, denote by $\mathcal{S}_k^{\text{new}}(N,\omega)$ the set of normalized newforms $f$ of weight $k$, level $N$ and character $\omega$, with Hecke eigenvalues $a_p(f)$, for all $p$ not dividing $N$, and corresponding $p-$Euler factors

$$L_p(s,f) = (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1},$$

where $\alpha_p = \alpha_p(f)$ and $\beta_p = \beta_p(f)$ are non-zero algebraic integers satisfying

$$a_p(f) = \alpha_p + \beta_p, \qquad \text{and} \qquad \omega(p)p^{k-1} = \alpha_p\beta_p.$$

Let us set

$$L_p(s,\mathrm{Ad(f)}) = (1-\frac{\alpha_\mathrm{p}}{\beta_\mathrm{p}}\mathrm{p}^{-\mathrm{s}})^{-1}(1-\mathrm{p}^{-\mathrm{s}})^{-2}(1-\frac{\beta_\mathrm{p}}{\alpha_\mathrm{p}}\mathrm{p}^{-\mathrm{s}})^{-1}.$$

**Theorem A** *Let $f \in \mathcal{S}_k^{\text{new}}(N,\omega)$ and $g \in \mathcal{S}_{k'}^{\text{new}}(N',\omega')$, $k \geqslant k'$, be such that, for all primes $p$ outside a set $S$ of Dirichlet density $\delta(S) < \frac{1}{18}$, we have*

$$(*) \qquad L_p(s,\mathrm{Ad(f)}) = \mathrm{L_p(s,Ad(g))}.$$

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*Then $k = k'$, and there exists a Dirichlet character $\chi$ of conductor $M$ dividing $NN'$ such that*

$$a_p(f) \;=\; a_p(g)\chi(p),$$

*all $p$ prime to $NN'$. In particular, $\omega = \omega'\chi^2$.*

*If $f, g$ are not of CM type and have weights $k, k' \geqslant 2$, then the same conclusion results if (*) is assumed to hold only for a set of primes of positive density.*

When $f$ and $g$ have the **same character**, we can deduce the stronger result below:

**Corollary** *Let $f \in \mathcal{S}_k^{\rm new}(N, \omega)$ and $g \in \mathcal{S}_k^{\rm new}(N', \omega)$ be such that, for all primes $p$ outside a set $S$ of density $\delta(S) < \frac{1}{18}$, we have*

$$a_p(f)^2 \;=\; a_p(g)^2,$$

*Then there exists a* **quadratic** *character $\chi$ of conductor $M$ dividing $NN'$ such that*

$$a_p(f) \;=\; a_p(g)\chi(p),$$

*for all $p$ not dividing $NN'$. Moreover, if $\omega = 1$ and $N$, $N'$* **square-free***, then $f = g$.*

*When $f, g$ are not of CM type and of weight $\geqslant 2$, we get the same conclusion assuming only that $\delta(S)$ is $< 1$.*

**Theorem A $\Longrightarrow$ Corollary**. The hypotheses imply that $(\alpha_p(f)/\beta_p(f)) + (\beta_p(f)/\alpha_p(f)) + 1$ equals $(\alpha_p(g)/\beta_p(g)) + (\beta_p(g)/\alpha_p(g)) + 1)$, for all $p$ outside $S$. It is then easy to see that $L_p(s, \mathrm{Ad(f)})$ equals $L_p(s, \mathrm{Ad(g)})$, for all such $p$. So we may apply the Theorem and deduce the existence of a $\chi$ such that $a_p(f) = a_p(g)\chi(p)$, for all $p$ prime to $NN'$. Comparing squares, we see that $\chi$ must be quadratic.

Next let $N, N'$ be square-free, and $\omega$ trivial. Suppose $\chi$ is non-trivial. Denote by $\pi$, $\pi'$ the cuspidal automorphic representations of $\mathrm{GL}(2, \mathbf{A}_{\mathbf{Q}})$ of trivial central character associated to $f$, $g$ respectively. Then, up to exchanging $f$ and $g$ if necessary, $N = N(\pi)$ must be $N(\pi' \otimes \chi)$, the conductor of $\pi' \otimes (\chi \circ \det)$. (Here we are identifying $\chi$ with the idèle class character of $\mathbf{Q}$ it defines.) Since $N' = N(\pi')$ is square-free, and since $\pi'$ has trivial central character, one sees easily from the description of local representations and their conductors in [Ge], p.73, that the $p-$component $\pi'_p$ must be the unramified special (Steinberg) representation at every prime $p$ dividing $N'$. One sees then, by using the same theorem (loc. cit.) that $\mathrm{ord}_p(N(\pi' \otimes \chi)) \geqslant 2$, for any $p$ dividing the conductor $M$ of $\chi$. Since $\mathbf{Q}$ has class number 1, there are no unramified characters $\chi$. In other words, $N = N(\pi' \otimes \chi)$ is not square-free, giving the desired contradiction. QED.

**Proof of Theorem A**. We will in fact give **two proofs**. We fix a prime $\ell$ not dividing $NN'$, and begin with the theorems of Deligne ([De], for $k \geqslant 3$), Eichler-Shimura ([Sh], for $k = 2$), and Deligne-Serre ([DS] for $k = 1$), giving the existence, for $h = f$ or $g$, of an irreducible, continuous representation

$$\sigma_\ell(h) : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell),$$

such that, for any prime $p$ not dividing $N\ell$,

$$\mathrm{tr}(\sigma_\ell(h)(Fr_p)) = a_p(h) = \alpha_p(h) + \beta_p(h), \quad |\alpha_p(h)| = |\beta_p(h)| = p^{(k(h)-1)/2},$$

and

$$\det(\sigma_\ell(h)) = \omega(h)\chi_{\mathrm{cyc}}^{k(h)-1}.$$

Here $Fr_p$ denotes the Frobenius conjugacy class at $p$, $\overline{\mathbf{Q}}_\ell$ a fixed algebraic closure of $\mathbf{Q}_\ell$, and $\chi_{\mathrm{cyc}}$ the cyclotomic character given by the Galois action on the inverse system of $\ell^m-$th roots of unity. ($\omega(h)$ is $\omega$ or $\omega'$ depending on whether $h$ is $f$ or $g$; similarly for $k(h)$.) If we consider the field $E$ generated by the coefficients of $f$, and a place $\lambda$ of $E$ above $\ell$, then one has in fact a representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\mathrm{GL}_2(E_\lambda)$, and our $\sigma_\ell$ is its extension to $\overline{\mathbf{Q}}_\ell$. We work over $\overline{\mathbf{Q}}_\ell$ because we will need to appeal to Schur's lemma.

For any two dimensional $\overline{\mathbf{Q}}_\ell-$representation $\sigma_\ell$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, set

$$\mathrm{Ad}(\sigma_\ell) = \mathrm{sym}^2(\sigma_\ell) \otimes \det(\sigma_\ell)^{-1}.$$

**Theorem B** *Let $K$ be a number field, and let $\sigma_\ell$ and $\sigma'_\ell$ be irreducible two dimensional $\overline{\mathbf{Q}}_\ell$-representations of Gal($\overline{\mathbf{Q}}/K$) with Frobenius traces $a_P$, $a'_P$ (for almost all primes $P$) and conductors $N$, $N'$ respectively. Suppose* $\mathrm{Ad}(\sigma_\ell) \simeq \mathrm{Ad}(\sigma'_\ell)$. *Then there exists $\psi_\ell \in Hom_{\mathrm{cont}}(\mathrm{Gal}(\overline{\mathbf{Q}}/K), \overline{\mathbf{Q}}_\ell^*)$ such that*

$$\sigma_\ell \simeq \sigma'_\ell \otimes \psi_\ell.$$

*Next let $K = \mathbf{Q}$. Suppose we know either that $\sigma_\ell$ and $\sigma'_\ell$ are Hodge-Tate (see [Se1]) or that the ratio of their determinants is a finite order character times an even power of $\chi_{\mathrm{cyc}}$. Then*

$$(**) \qquad \psi_\ell = \chi_{\mathrm{cyc}}^r \nu_\ell,$$

*where $r$ is an integer, and $\nu_\ell$ the $\ell-$adic character defined by a Dirichlet character $\nu$.*

**Theorem B $\Longrightarrow$ Theorem A**. Let $f$, $g$ be as in Theorem A. Since $\sigma_\ell(f)$ and $\sigma_\ell(g)$ are simple, $\mathrm{Ad}(\sigma_\ell(\mathrm{f}))$ and $\mathrm{Ad}(\sigma_\ell(\mathrm{g}))$ are semisimple, and we claim that they are isomorphic.

Modulo this claim, we proceed as follows. Applying the first part of Theorem B, we get a character $\psi_\ell$ such that $\sigma_\ell(f) \simeq \sigma_\ell(g) \otimes \psi_\ell$. Comparing determinants, we get for almost all $p$,

$$(I) \qquad \psi_\ell(Fr_p)^2 = \chi_{\mathrm{cyc}}(Fr_p)^{k-k'}\omega(p)\omega'(p)^{-1}.$$

At this point, one can use (at least) three different methods to finish the argument. The first uses a theorem of Faltings [Fa], which says that $\sigma_\ell(h)$ is Hodge-Tate for any newform $h$ of conductor prime to $\ell$. So, by the second part of Theorem B, $k$ and $k'$ are of the same parity, and we get (**) with $r = (k-k')/2$. Let $H(f)$ (resp. $H(g)$) be the $\mathbf{Q}$-Hodge structure of weight $k-1$ (resp. $k'-1$) associated to (the motive of) $f$ (resp. $g$). Then we must have $H(f) \simeq H(g)(r)$, where $H(g)(r)$ denotes the Tate twist $H(g) \otimes \mathbf{Q}(r)$. Then $r$ must be zero, since the Hodge type of $H(f)$ (resp. $H(g)$) is $\{(k-1,0),(0,k-1)\}$ (resp. $\{(k'-1,0),(0,k'-1)\}$), while that of $H(g)(r)$ is $\{(k'-1-r,-r),(-r,k'-1-r)\}$. Done.

The second method uses $L$-functions. Let $\nu$ be the finite order character defined as $\psi_\ell \chi_{\rm cyc}^{(k'-k)/2}$. Then by (I) we have, for every Dirichlet character $\mu$, an identity

$$L_p(s, f\otimes\mu) \;=\; L_p(s-(k-k')/2, g\otimes\mu\nu),$$

for all $p$ in the set $T$ of all primes not dividing $\ell N N'$ and the conductor of $\mu$. We may fix a $\mu$, sufficiently ramified at the primes in $T$, such that the local factors of $f\otimes\mu$ and $g\otimes\nu\mu$ at any prime in $T$ are 1. Interchanging $f$ and $g$ if necessary, we may assume that $k \leqslant k'$. Since the archimedean factor attached to $f\otimes\mu$ is $(2\pi)^{-s}\Gamma(s)$, and since its product with (the global Euler product) $L(s, f\otimes\mu)$ is entire, any pole of the Gamma factor results in a zero of $L(s, f\otimes\mu)$, which is $\prod_{p\notin T} L_p(s, f\otimes\mu)$ by the choice of $\mu$. This happens for example at $s=0$, and consequently, by the identity above, $L(s+(k'-k)/2)$ has a zero at $s=0$, even though its the archimedean factor does not have a pole there (as $k'>k$). Then, by applying the functional equation for $g\otimes\mu\nu$ (which relates $s$ to $k'-s$), we see that $L(s,\overline{g}\otimes\overline{\mu\nu})$ has a zero at $s=(k'+k)/2$. This is absurd (see [JS]) as this point is in the region (resp. on the boundary) of absolute convergence if $k>1$ (resp. $k=1$). So we must have $k=k'$.

The third method is to appeal, for $\ell$ large enough, to the mod $\ell$ result proved later in this appendix.

Now we prove the claim. The identity (*) says that the characteristic polynomials of the Frobenius classes $Fr_p$ agree on $\mathrm{Ad}(\sigma_\ell(\mathrm{f}))$ and $\mathrm{Ad}(\sigma_\ell(\mathrm{g}))$, for all $p$ outside a set $S$ of density $\delta < \frac{1}{18}$. If $\delta(S)=0$, then by the Tchebotarev density theorem, $\mathrm{Ad}(\sigma_\ell(\mathrm{f}))$ and $\mathrm{Ad}(\sigma_\ell(\mathrm{g}))$ would be equivalent, and our object is to get the same conclusion under the weaker hypothesis on $\delta$. By [GJ], we know that, for $h=f$ or $g$, there is an (isobaric) automorphic representation $\mathrm{Ad}(\mathrm{h})$ of $\mathrm{GL}(3,\mathbf{A}_\mathbf{Q})$, whose standard $L$-function identifies, after removing the archimedean factors, with $\prod_p L_p(s-1,\mathrm{Ad}(\mathrm{h}))$. It suffices to show that $\mathrm{Ad}(\mathrm{f})$ and $\mathrm{Ad}(\mathrm{g})$ are isomorphic. Suppose not. Then we can find (isobaric) automorphic representations $\pi,\pi'$ of $\mathrm{GL}(k,\mathbf{A}_\mathbf{Q})$, $k\leqslant 3$, such that $\mathrm{Ad}(\mathrm{f}) \simeq \pi\boxplus\eta$ and $\mathrm{Ad}(\mathrm{g}) \simeq \pi'\boxplus\eta$, where $\eta$ is an automorphic representation of $\mathrm{GL}(3-k,\mathbf{A}_\mathbf{Q})$, taken to be 0 if $k=3$. Let $Z_S(s)$ be as in equation (3) of [Ra]. In the present case, if $m$ (resp. $r$) denotes the number of cuspidals occurring in the isobaric decomposition [La] of $\pi$ (resp. $\pi'$), necessarily with multiplicity 1, we have $-\mathrm{ord}_{s=1} Z_S(s) = m^2+r^2$ (compare with (4) of [Ra]). Since one knows the Ramanujan conjecture for holomorphic forms by Deligne, it is easy to verify that

Lemma 2 of [Ra] holds for $\pi$ (resp. $\pi'$) with $\beta$ less than $k^2m^2\delta$ (resp. $k^2r^2\delta$). Then the argument of section 2 of [Ra] shows that we must have $1 \leqslant 2k^2\delta$. Since $\delta < 1/18$ and $k \leqslant 3$, we get the desired contradictiction.

It remains to treat the case when $f, g$ are not of CM type and have weights $\geqslant 2$, with $\delta$ assumed to be just $< 1$. One knows by the works of Serre and Ribet [Ri] that $\sigma_\ell(f)$ is absolutely irreducible under restriction to any open subgroup. We note then that the same must be true for $\mathrm{Ad}(\sigma_\ell(\mathrm{f}))$, as otherwise the restriction $\sigma_\ell(f)_K$ will, for some number field $K$, be induced by a character of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$, for a quadratic extension $F/K$ (see below), making $\sigma_\ell(f)_F$ reducible. Now, applying Theorem 2 of [Raj] for example, we may conclude that, as $\delta < 1$, $\mathrm{Ad}(\sigma_\ell(\mathrm{f}))$ must be isomorphic to $\mathrm{Ad}(\sigma_\ell(\mathrm{g})) \otimes \nu_\ell$, for some one-dimensional $\nu_\ell$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ defined by a Dirichlet character. Let $K$ be the cyclic extension of $\mathbf{Q}$ corresponding to $\nu_\ell$, and let $\tau$ be a generator of $\mathrm{Gal}(K/\mathbf{Q})$. Then, since $\mathrm{Ad}(\sigma_\ell(\mathrm{f})_\mathrm{K})$ and $\mathrm{Ad}(\sigma_\ell(\mathrm{g})_\mathrm{K})$ are isomorphic, we may apply Theorem B and conclude that $\sigma_\ell(f)_K \simeq \sigma_\ell(g)_K \otimes \lambda_\ell$, for a character $\lambda_\ell$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$. Since $\sigma_\ell(f)_K$ and $\sigma_\ell(g)_K$ are invariant under $\tau$, we get

$$\sigma_\ell(g)_K \otimes (\lambda/\lambda^{[\tau]}) \simeq \sigma_\ell(g)_K.$$

Since $\sigma_\ell(g)$ is irreducible under restriction to any open subgroup, $\sigma_\ell(g)_K$ cannot admit any non-trivial self-twist, and $\lambda$ must be invariant under $\tau$ and hence must extend to a character of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The rest of the argument goes through as above, and Theorem A follows.

**Proof of Theorem B**. First we need a simple

**Lemma**. *Let $\rho_\ell$ be an irreducible, $n-$dimensional, self-dual $\overline{\mathbf{Q}}_\ell$-representation of Gal$(\overline{\mathbf{Q}}/K)$. Then there exists an invariant non-degenerate bilinear form $B$ on (the space of) $\rho_\ell$, which is symmetric or alternating, such that*

- *(i) $B$ is unique up to a non-zero scalar; and*
- *(ii) If $\rho'_\ell$ is another irreducible, $n-$dimensional, self-dual $\overline{\mathbf{Q}}_\ell$-representation of Gal$(\overline{\mathbf{Q}}/K)$ with invariant non-degenerate bilinear form $B'$, such that $\rho_\ell$ and $\rho'_\ell$ are isomorphic, then they are* **isometric** *relative to $B$ and $B'$.*

Indeed, (i) and the statement above it are immediate consequences of Schur's lemma. Also, since $\overline{\mathbf{Q}}_\ell$ is algebraically closed, $cB$ is isometric to $B$ for any $c \in \overline{\mathbf{Q}}_\ell^*$; hence we get (ii) as well.

Now let $\sigma_\ell$ and $\sigma'_\ell$ be as in Theorem B. Suppose (the semisimple representation) $\mathrm{Ad}(\sigma_\ell)$ is reducible. Then it must contain a one dimensional summand $\eta_\ell$, say. Then $\eta_\ell$ occurs in the (self-dual) $\mathrm{End}(\sigma_\ell) = \sigma_\ell \otimes \sigma_\ell^\vee = \mathrm{Ad}(\sigma_\ell) \oplus 1$. Schur's lemma above forces $\eta_\ell$ to be non-trivial. Either $\eta_\ell$ is quadratic, or otherwise $\eta_\ell^\vee$ will also occur in $\mathrm{End}(\sigma_\ell)$. In either case, we see that $\mathrm{End}(\sigma_\ell)$ must contain a quadratic character $\delta_\ell$, say; let $F$ be the corresponding quadratic extension of $K$ with non-trivial automorphism $\theta$. Denote by $\sigma_{F,\ell}$ the restriction of $\sigma_\ell$ to $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$. We claim (as is well known) that if $\tau_\ell$ is another semisimple representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ whose restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ is isomorphic to $\sigma_{F,\ell}$, then $\tau_\ell \simeq \sigma_\ell \otimes \delta_\ell^j$, for $j \in \{0,1\}$. Indeed, by the hypothesis, the restriction of $\eta_\ell := \tau_\ell \otimes \sigma_\ell^\vee$ to $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ contains the trivial representation; so by Frobenius reciprocity, there is a non-trivial homomorphism between $\eta_\ell$ and the representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ induced by the trivial representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$, which decomposes as $1 \oplus \delta_\ell$. So $\delta_\ell^j$ occurs in $\eta_\ell$, for $j = 0$ or 1. Equivalently, there is an intertwining operator between $\tau_\ell$ and $\sigma_\ell \otimes \delta_\ell^j$, which implies the claim by virtue of the irreducibility of $\sigma_\ell$. Next observe that $\sigma_{F,\ell}$ must be reducible as $\mathrm{End}(\sigma_{F,\ell})$ contains 1 with multiplicity 2 (as the restriction of $\delta_\ell$ to $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ is trivial). Write $\sigma_{F,\ell} = \nu_\ell \oplus \mu_\ell$, with $\nu_\ell, \mu_\ell$ being one-dimensionals of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$. We claim that $\nu_\ell$ is not $\theta$-invariant. Indeed, otherwise $\mu_\ell$ would also be $\theta$-invariant as $\sigma_{F,\ell}$ is, and both $\nu_\ell$ and $\mu_\ell$ would admit extensions to $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ and result in a reducible extension of $\sigma_{\ell,F}$, which is impossible by the claim above. Thus $\nu_\ell$ is not fixed by $\theta$, and so we must have $\sigma_{F,\ell} \simeq \nu_\ell \oplus \nu_\ell^{[\theta]}$. This forces $\sigma_\ell$ to be the induced representation $\mathrm{Ind}_F^K(\nu_\ell)$, as this induced representation has the same restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ as $\sigma_\ell$ and is moreover isomorphic to its twist by any character of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ trivial on $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$. Since $\mathrm{End}(\sigma_\ell) = \mathrm{End}(\sigma'_\ell)$, $\sigma'_\ell$ must also be of the form $\mathrm{Ind}_F^K(\nu'_\ell)$, for some one-dimensional $\nu'_\ell$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$. Since the determinant of $\mathrm{Ind}_F^K(\nu_\ell)$ is the transfer of $\nu_\ell$ to $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ times $\delta_\ell$, we see that

$$\mathrm{Ad}(\sigma_\ell) \simeq \mathrm{Ind}_{\mathrm{F}}^{\mathrm{K}}(\nu_\ell/\nu_\ell^{[\theta]}) \oplus \delta_\ell,$$

and similarly for $\mathrm{Ad}(\sigma'_\ell)$. This implies that, up to replacing $\nu_\ell$ by $\nu_\ell^{[\theta]}$, we have

$$\nu_\ell/\nu_\ell^{[\theta]} = \nu'_\ell/(\nu'_\ell)^{[\theta]}.$$

Then $\nu_\ell/\nu'_\ell$ is $\theta-$invariant, and hence extends to a character $\psi_\ell$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$. In other words, $\sigma_\ell \simeq \sigma'_\ell \otimes \psi_\ell$, as claimed.

We next consider the case when $\mathrm{Ad}(\sigma_\ell)$ and $\mathrm{Ad}(\sigma'_\ell)$ are irreducible. Let $\lambda_\ell$ denote the product of the determinants $\omega_\ell$, $\omega'_\ell$ of $\sigma_\ell$, $\sigma'_\ell$ respectively. Set

$$\eta_\ell := \sigma_\ell \otimes \sigma'_\ell.$$

Then

$$\mathrm{sym}^2(\eta_\ell) \otimes \lambda_\ell^{-1} \simeq \mathrm{Ad}(\sigma_\ell) \otimes \mathrm{Ad}(\sigma'_\ell) \oplus 1.$$

Since $\mathrm{Ad}(\sigma_\ell)$ and $\mathrm{Ad}(\sigma'_\ell)$ are irreducible, self-dual and isomorphic, 1 occurs in their tensor product. Hence the multiplicity of $\lambda_\ell$ is greater than 1 in $\mathrm{sym}^2(\eta_\ell)$, showing that $\eta_\ell$ is reducible. Now suppose $\eta_\ell$ contains a two dimensional summand $\tau_\ell$, say. Then the one dimensional $\det(\tau_\ell)$ occurs in the exterior square of $\eta_\ell$. But on the other hand, we have

$$\Lambda^2(\eta_\ell) \simeq \mathrm{sym}^2(\sigma_\ell) \otimes \omega'_\ell \oplus \omega'_\ell \otimes \mathrm{sym}^2(\sigma'_\ell),$$

showing that, as the symmetric squares of $\sigma_\ell$ and $\sigma'_\ell$ are irreducible, there can be no one dimensional summand of $\Lambda^2(\eta_\ell)$. This shows that $\eta_\ell$ has no two dimensional summand. Since it is reducible, it must then have a one dimensional summand $\nu_\ell$, say. Then

$$\sigma_\ell \simeq {\sigma'_\ell}^\vee \otimes \nu_\ell \simeq \sigma'_\ell \otimes {\omega'_\ell}^{-1}\nu_\ell.$$

So we get the desired $\psi_\ell$ by taking it to be ${\omega'_\ell}^{-1}\nu_\ell$.

Now let $K = \mathbf{Q}$. Comparing determinants, we see that $\psi_\ell^2 = \det(\sigma_\ell)\det(\sigma'_\ell)^{-1}$. So we get (**) immediately if the ratio of the determinants is a finite order character times an even power of $\chi_{\rm cyc}$. Finally, suppose $\sigma_\ell$ and $\sigma'_\ell$ are Hodge-Tate. Then $\psi_\ell$ will also be Hodge-Tate as it occurs in $\sigma_\ell \otimes (\sigma'_\ell)^\vee$. Consequently, it corresponds to an algebraic Hecke character $\psi$. Since we are working over $\mathbf{Q}$, it must be a finite order character times a power of $\chi_{\rm cyc}$. Done.

For the second proof, we begin by recalling the fact that the adjoint representation Ad: $\mathrm{PGL}(2, \overline{\mathbf{Q}}_\ell) \longrightarrow \mathrm{GL}(3, \overline{\mathbf{Q}}_\ell)$ is isomorphic onto the special orthogonal group $\mathrm{SO}(3, \overline{\mathbf{Q}}_\ell)$. Denote by $\overline{\sigma}_\ell$ (resp. $\overline{\sigma}'_\ell$) the composite of $\sigma_\ell$ (resp. $\sigma'_\ell$) with the natural homomorphism of $\mathrm{GL}(2, \overline{\mathbf{Q}}_\ell)$ onto $\mathrm{PGL}(2, \overline{\mathbf{Q}}_\ell)$. Then it is easy to see that $\mathrm{Ad}(\overline{\sigma}_\ell)$ identifies with the $\mathrm{Ad}(\sigma_\ell)$ defined earlier (above Theorem B). So, by our hypothesis, we get two representations, namely $\mathrm{Ad}(\overline{\sigma}_\ell)$ and $\mathrm{Ad}(\overline{\sigma}'_\ell)$, into $\mathrm{SO}(3, \overline{\mathbf{Q}}_\ell)$, which are equivalent in $\mathrm{GL}(3, \overline{\mathbf{Q}}_\ell)$. Suppose they are irreducible. Then we may apply part (ii) of the Lemma and deduce that they are in fact isometric. By changing the isometry by $-I$ if necessary, we may assume that they are equivalent in $\mathrm{SO}(3, \overline{\mathbf{Q}}_\ell)$. Since Ad is an isomorphism, $\sigma_\ell$ and $\sigma'_\ell$ define equivalent homomorphisms into $\mathrm{PGL}(2, \overline{\mathbf{Q}}_\ell)$. Hence $\sigma_\ell$ must be equivalent to $\sigma'_\ell \otimes \psi_\ell$, for some $\psi_\ell \in \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbf{Q}}/K), \overline{\mathbf{Q}}_\ell^*)$. When $\mathrm{Ad}(\sigma_\ell)$ is reducible, one uses explicit arguments as in the reducible case of the first proof to conclude that $\mathrm{Ad}(\sigma_\ell)$ and $\mathrm{Ad}(\sigma'_\ell)$ are isometric. The rest follows.
QED.

**The mod $\ell$ version**. For each newform $f$, let $K_f$ denote the number field generated by the coefficients of $f$. If $g$ is another newform, let $\mathfrak{O}_{f,g}$ denote the ring of integers of the compositum $K_fK_g$. For $h = f$ or $g$, write for $p$ not dividing the level,

$$Q_h(T) = (1 - \frac{\alpha_p(h)}{\beta_p(h)}T)(1-T)(1 - \frac{\beta_p(h)}{\alpha_p(h)}T),$$

so that $L_p(s, \mathrm{Ad(h)}) = \mathrm{Q_h(p^{-s})^{-1}}$. Note that, since $\alpha_p(h)\beta_p(h) = \omega(h)p^{k(h)-1}$, $\alpha_p(h)$ and $\beta_p(h)$ are invertible modulo any prime $\ell$ not dividing $pN(h)$.

**Theorem C** *Let $\ell$ be an odd prime number and $N, N'$ positive integers prime to $\ell$. Let $f$ (resp. $g$) be a newform of level $N$ (resp. $N'$), weight $k$ (resp. $k'$), and character $\omega$ (resp. $\omega'$). Let $\lambda$ be a prime ideal above $\ell$ in $\mathfrak{O}_{f,g}$. Suppose we have*

$$(C) \qquad Q_f(T) \equiv Q_g(T) \pmod{\lambda},$$

*for all $p$ outside a set $S$ (containing the primes divisors of $\ell NN'$) of density* $0$. *Then $k \equiv k'$ (mod $\ell - 1$), and there exists a character $\beta$, unramified at $\ell$, such that*

$$a_p \equiv b_p\beta(p) \pmod{\lambda},$$

*for all $p$ not dividing $\ell NN'$.*

**Remark**: Note that if $\omega$ and $\omega'$ are the same mod $\lambda$, and if $k - k' \equiv 0$(mod $\ell - 1$), the hypothesis (C) is equivalent to the congruence

$$a_p^2 \equiv b_p^2 \pmod{\lambda}.$$

In this case $\beta$ is necessarily quadratic. Moreover, if $N$ and $N'$ are in addition square-free, one can conclude (as in the characteristic zero case) that $\beta$ is trivial.

*Proof.* Let $\mathbf{F}_\lambda$ denote the residue field $\mathfrak{O}_{f,g}/\lambda$. Reducing the (integrally defined) $\ell$-adic representations associated to $f, g$ modulo $\lambda$ and extending scalars to $\overline{\mathbf{F}}_\lambda$, we get representations

$$\overline{\sigma}_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_\lambda)$$

and

$$\overline{\sigma}'_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_\lambda)$$

such that, for all $p$ not dividing $NN'\ell$, $\mathrm{tr}(\overline{\sigma}_\lambda(\mathrm{Fr}_p))$ (resp. $\mathrm{tr}(\overline{\sigma}'_\lambda(\mathrm{Fr}_p))$) is the image of $a_p$ (resp. $b_p$) in $\overline{\mathbf{F}}_\lambda$. Moreover, by hypothesis, $\det(\overline{\sigma}_\lambda)$ and $\det(\overline{\sigma}'_\lambda)$ both equal $\chi^{k-1}\overline{\omega}$ (resp. $\chi^{k'-1}\overline{\omega}'$), where $\chi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{F}_\ell^*$ is the mod $\ell$ cyclotomic character and $\overline{\omega}$ (resp. $\overline{\omega}'$) the reduction (mod $\lambda$) of $\omega$ (resp. $\omega'$). Clearly, the images of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ under these two representations are finite.

For any $\overline{\mathbf{F}}_\lambda$-representation $\tau_\lambda$ of a finite group $G$ of dimension $d$, let $\tau_\lambda^{\mathrm{ss}}$ denote its semisimplification. Note that in characteristic $\ell$, the semisimplification is determined by the characteristic polynomials of $\tau_\lambda(g)$ for all $g$ in $G$ when $d > \ell$, and also when $d = \ell = 3$ if $\tau_\lambda$ is orthogonal of determinant 1.

By the hypothesis (C), the characteristic polynomials of $\mathrm{Fr}_p$ in the adjoint representations of $\overline{\sigma}_\lambda$ and $\overline{\sigma}'_\lambda$ are the same for all $p$ in a set of density 1. Thus, by the Tchebotarev density theorem and the remark above, we see that

$$\mathrm{Ad}(\overline{\sigma}_\lambda^{\mathrm{ss}}) \simeq \mathrm{Ad}(\overline{\sigma}_\lambda'^{\,\mathrm{ss}}).$$

Since $\mathrm{End}(\overline{\sigma}_\lambda^{\mathrm{ss}})$ (resp. $\mathrm{End}(\overline{\sigma}_\lambda'^{\,\mathrm{ss}})$) is $\mathrm{Ad}(\overline{\sigma}_\lambda^{\mathrm{ss}}) \oplus 1$ (resp. $\mathrm{Ad}(\overline{\sigma}_\lambda'^{\,\mathrm{ss}}) \oplus 1$), it follows that $\overline{\sigma}_\lambda$ is irreducible iff $\overline{\sigma}'_\lambda$ is.

First suppose that $\overline{\sigma}_\lambda$ and $\overline{\sigma}'_\lambda$ are irreducible. In this case the detailed $\ell$-adic argument given in the proof of (the first part of) Theorem B goes through, with $\overline{\mathbf{Q}}_\ell$ replaced everywhere by $\overline{\mathbf{F}}_\lambda$, once one notes the availability of the relevant form of the Frobenius reciprocity in characteristic $\ell$ (cf. [A], chap. III, Lemma 6) and the fact that the tensor square of a simple Galois module is semisimple [Se3]. One deduces an isomorphism of $\overline{\sigma}_\lambda$ with $\overline{\sigma}'_\lambda \otimes \nu_\lambda$, for some character $\nu_\lambda$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\overline{\mathbf{F}}_\lambda$. Since $\omega_\lambda$ and $\omega'_\lambda$ are the same modulo $\lambda$, we see by comparing determinants that $\nu_\lambda^2$ is $\chi^{k-k'}\overline{\omega}/\overline{\omega}'$. We may write $\nu_\lambda$ as $\chi^j\beta_\lambda$, for some $j \in \{0, \ldots, \ell-2\}$, and a character $\beta_\lambda$ unramified at $\ell$. Consequently, $k - k' \equiv 2j(\bmod \ell - 1)$, $\beta_\lambda^2 = \overline{\omega}/\overline{\omega}'$, and

$$(***) \qquad \overline{\sigma}_\lambda \simeq \overline{\sigma}'_\lambda \otimes \chi^j \beta_\lambda.$$

Let $G_\ell$ denote the decomposition group at $\ell$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and let $I$ denote the inertia subgroup. When $a_\ell$ is not zero modulo $\lambda$, one knows by Deligne (cf. [E], Theorem 2.5, for example), that $\overline{\rho}_\lambda|_{G_\ell}$ is reducible, and its semisimplification is of the form $\chi^{k-1}\mu_{1,\lambda} \oplus \mu_{2,\lambda}$, where each $\mu_{j,\lambda}$ is unramified. When $a_\ell$ is divisible by $\lambda$, a result of Fontaine (see [E], Theorem 2.6) asserts that the restriction to $G_\ell$ is irreducible, while the restriction to $I$ demposes as $\psi^{k-1} \oplus \psi'^{k-1}$, where $\psi, \psi'$ are the two fundamental characters of level 2 [Se2]. Similarly for the restriction of $\overline{\sigma}'_\lambda$ at $\ell$. In either case, we see that the only way (***) can hold is for $j$ to be 0 modulo $\ell - 1$.

It remains to consider when $\overline{\sigma}_\lambda$ (and hence $\overline{\sigma}'_\lambda$) is reducible. Here we may write

$$\overline{\sigma}_\lambda^{\mathrm{ss}} \simeq \eta_\lambda \oplus \chi^{k-1}\overline{\omega}/\eta_\lambda,$$

and

$$\overline{\sigma}'^{\,\mathrm{ss}}_\lambda \simeq \eta'_\lambda \oplus \chi^{k'-1}\overline{\omega}/\eta'_\lambda,$$

for some $\overline{\mathbf{F}}_\lambda^*$-valued characters $\eta_\lambda, \eta'_\lambda$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Then we have

$$\mathrm{Ad}(\overline{\sigma}_\lambda^{\mathrm{ss}}) \simeq \eta_\lambda^2/\overline{\omega}\chi^{k-1} \oplus 1 \oplus \overline{\omega}\chi^{k-1}/\eta_\lambda^2,$$

and

$$\mathrm{Ad}(\overline{\sigma}'^{\,\mathrm{ss}}_\lambda) \simeq {\eta'_\lambda}^2/\overline{\omega}\chi^{k'-1} \oplus 1 \oplus \overline{\omega}\chi^{k'-1}/{\eta'_\lambda}^2.$$

Since Ad commutes with semisimplification, it follows, after possibly replacing $\eta_\lambda$ with $\chi^{k-1}\overline{\omega}/\eta_\lambda$, that $\eta_\lambda^2/\chi^k = {\eta'_\lambda}^2/\chi^{k'}$. Arguing as above, we see that $\eta_\lambda$ is of the form $\eta'_\lambda\chi^j\beta_\lambda$, for some $j \in \{0, \ldots, \ell-2\}$ with $k - k' \equiv 2j(\bmod \ell - 1)$, and a character $\beta_\lambda$ :$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \overline{\mathbf{F}}_\lambda^*$, unramified at $\ell$, such that $\beta_\lambda^2 = \overline{\omega}/\overline{\omega}'$. We obtain

$$\overline{\sigma}_\lambda^{\,\mathrm{ss}} \simeq \eta'_\lambda\beta_\lambda\chi^{(k-k')/2} \oplus \beta_\lambda\chi^{(k+k')/2-1}\overline{\omega}'/\eta'_\lambda.$$

The reducibility of $\overline{\sigma}_\lambda$ (resp. $\overline{\sigma}'_\lambda$) forces $a_\ell$ (resp. $b_\ell$) to be non-zero modulo $\lambda$, as the restriction of $\overline{\sigma}_\lambda^{ss}$ (resp. $\overline{\sigma}'^{\,ss}_\lambda$) to $I$ must then be given by a direct sum of characters of level 1 [Se2]. Applying Deligne's result on the shape of the restriction to $G_\ell$ (see above), we see that the only possibility is for $k$ and $k'$ to be congruent modulo $\ell - 1$. Then $\overline{\sigma}_\lambda^{ss}$ is isomorphic to $\overline{\sigma}'^{\,ss}_\lambda \otimes \beta_\lambda$. Done.

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