SELF-DUAL REPRESENTATIONS OF DIVISION ALGEBRAS AND WEIL GROUPS: A CONTRAST

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Introduction

If ρ is a selfdual representation of a group G on a vector space V over $\mathbb{C}$, we will say that ρ is *orthogonal*, resp. *symplectic*, if G leaves a nondegenerate *symmetric*, resp. *alternating*, bilinear form B on V invariant. If ρ is irreducible, exactly one of these possibilities will occur, with B unique up to scaling, and we may define a *sign* $c(\rho) \in \{\pm 1\}$, taken to be $+1$, resp. -1, in the orthogonal, resp. symplectic, case.

Now let k be a local field of characteristic 0. The groups of interest to us will be $G = \mathrm{GL}_m(D)$, where D is a division algebra of center k and index r, and the Weil group W_k. Note that the group $\mathrm{GL}_1(D)$ $(= D^\times)$ is even compact modulo the center, hence its complex irreducible representations π are finite dimensional. For any m, let $R^0(G)$ denote the set of irreducible admissible $\mathbb{C}$-representations π (up to equivalence) which correspond, by the generalized Jacquet-Langlands correspondence ([Bad]), to irreducible discrete series representations π' of $\mathrm{GL}_n(k)$, with $n = mr$. When used in conjunction with the local Langlands correspondence ([HT], [Hen1]), there exists a bijection $\pi \mapsto \sigma$ satisfying certain natural properties, such as the preservation of ε-factors of pairs, between $R^0(G)$ and the set $Irr_n(W_k')$ of irreducible representations σ of W_k' of dimension n, again taken up to equivalence. Here W_k' denotes W_k if k is archimedean, and the extended Weil group $W_k \times \mathrm{SL}(2,\mathbb{C})$ if k is non-archimedean. One calls σ the *Langlands parameter* of π. It is immediate from the construction that π is selfdual if and only if σ is. However, the local Langlands reciprocity is not *a priori* sensitive to the *finer question* of whether $c(\pi)$ equals $c(\sigma)$ or $-c(\sigma)$. The main result of this paper is the following.

Theorem A *Let $n = mr$, D a division algebra of index r over a local field k of characteristic zero, $G = \mathrm{GL}_m(D)$, and π a selfdual representation of G in $R^0(G)$ with parameter $\sigma \in Irr_n(W_k')$. Then we have*

$$(-1)^m c(\pi) \,=\, (-1)^n c(\sigma)^m.$$

Corollary B *Let π be an irreducible, selfdual representation of $D^\times$, for any division algebra of index n over a local field k of characteristic zero, and let σ be the n-dimensional parameter of π. If n is odd, π is always orthogonal, while if n is even,*

$$\pi \quad \text{orthogonal} \quad \Longleftrightarrow \quad \sigma \quad \text{symplectic}.$$

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When n is odd, σ is necessarily orthogonal, and so Theorem A implies, for any factorization $n = mr$ and $G = \mathrm{GL}_m(D)$ with D a division algebra of index r, that $c(\pi) = +1$, i.e., π is orthogonal like its parameter. Now let $m = 1$. Then we get $c(\pi) = (-1)^{n+1}c(\sigma)$, which implies that *for n even, π is symplectic if and only if σ is odd*, proving the Corollary, assuming the Theorem. This surprising *flip* for n even is what we noticed first for $n = 2$, spurring our interest in the general case, which is more subtle to establish. Based on considerations of Poincaré duality on the middle dimensional cohomology of certain coverings of the Drinfeld upper-half space, we conjectured in [PR] the assertion of Corollary B, and established some positive results in [PR] and [Pra2], including the case of $n = 2$. In [Pra2] it was proved that if n is odd and if the residual characteristic of k is odd, then $D^\times$ has no selfdual irreducible representations of dimension > 1, showing that in this case, the conjecture is difficult only for the even residual characteristic. An engaging program to prove Corollary B along the geometric lines, using cohomological methods involving the formal moduli of Lubin-Tate groups, has been announced in the supercuspidal case by Laurent Fargues; it does not seem, however, that, without further input, his suggested methods would work for general discrete series representations, nor for $\mathrm{GL}_m(D)$ for $m > 1$. It may also be useful to note that, when $r = 1$, so that $G = \mathrm{GL}_n(k)$, the formula of Theorem A is easily seen to yield $c(\pi) = +1$, which is elementary to prove directly, but is nevertheless useful while using global arguments to deal with more difficult cases.

Let n be even and $m = 1$. Then Corollary B associates, to each irreducible, symplectic Galois representation σ of dimension n, a *new secondary invariant*, defined by whether, or not, the associated orthogonal representation π of $D^\times$ lifts to the (s)pin group. This aspect was investigated for $n = 2$ in [PR].

Our proof of Theorem A for non-archimedean k proceeds by using *global methods*, and the philosophy is that if the sign $c(\pi)$ is the asserted one for local representations π of simple type at one prime, then the sign should also be right for any complicated local representation at a *different* prime. This is why section 4 of the paper is spent verifying the assertion in a special case, namely when $G = D^\times$, n is even, and π is of *level* 1, i.e., trivial on the level 1 subgroup $D^\times(1)$. When the parameter is not trivial on $\mathrm{SL}(2, \mathbb{C})$, we have to use a variant of the global approach, having to deal (in the process) with generalized Steinberg representations of $\mathrm{GL}(n, k)$. When π is not of this type, we globalize appropriately and get a level 1 representation at another prime. When the associated representation of $\mathrm{GL}(n, k)$ is supercuspidal, We use the strong functoriality transfer between generic automorphic representations of classical groups and those of the corresponding general linear group admitting a pole of an appropriate L-function, which have been established in a series of papers due to many people, namely [CKPSS], [CPSS], [JS1], [GJR], [JS2]). A synthesis of these results is a necessary first step in proving Theorem A for $G = D^\times$, and even there the case when the parameter is non-trivial on $\mathrm{SL}(2, \mathbb{C})$ requires a new twist, involving a different sort of globalization (see section 3), which allows us to transport the sign problem to one with trivial $\mathrm{SL}(2)$-parameter. We then use the result in this crucial case to attack the general situation of $G = \mathrm{GL}_m(D)$ later.

The workhorse which allows us to use global methods is the following *product formula*, reminiscent of quadratic reciprocity.

Theorem C *Let F be a global field, $G = GL_m(\mathbb{D})$, where $\mathbb{D}$ is a division algebra of dimension r^2 over F and Z the center of G. Suppose $\Pi = \otimes'_v \Pi_v$ is an irreducible, selfdual automorphic representation of $G(\mathbb{A}_F)$ of central character ω, which occurs with multiplicity one in $L^2(G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F), \omega)$. Then we have*

$$c(\Pi) \;=\; \prod_{v \in \mathrm{ram}(\mathbb{D})} c(\Pi_v) \;=\; 1,$$

where ram($\mathbb{D}$) denotes the set of places where $\mathbb{D}$ is ramified.

As a consequence, we see that in the case $m = 1$ and $\mathrm{ram}(\mathbb{D}) = \{u, v\}$, we have $c(\Pi_u) = c(\Pi_v)$, i.e., Π_u and Π_v are both orthogonal or both symplectic. In particular, if we know one, we know the other.

There are two reasons why Theorem C holds. The first is that the *global sign* $c(\pi)$ is 1, and this works in greater generality by a simple argument (see section 2). The second is that the local sign $c(\pi_v)$ is 1 whenever $G(F_v)$ is $\mathrm{GL}_n(F_v)$.

Thanks to the product formula, given an irreducible selfdual representation π of $D^\times$, with D a division algebra over a local field k, our strategy becomes one of trying to find a number field F with $F_v = k$ for a place v of F, a division algebra $\mathbb{D}$ over F with $\mathrm{ram}(\mathbb{D}) = \{u, v\}$, and a *selfdual* automorphic representation Π of $\mathbb{D}^\times(\mathbb{A}_F)$ such that $\Pi_v = \pi$ and Π_u a known representation. Clearly, the parameters of π and Π_u must have the same parity. Thanks to the generalized Jacquet-Langlands correspondence between $\mathbb{D}^\times$ and GL_n/F, we see that it suffices to find a *selfdual* discrete automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_F)$ with Π'_v, resp. Π'_u, being associated to Π_v, resp. Π_u. The difficulty is not in globalizing, but in choosing a global Π' which is also selfdual. We have, thankfully, been able to address this, when Π_u and Π_v are both supercuspidal by using various known instances of Functoriality, culminating in the works of Jiang and Soudry ([JS2], [Sou1]).

The statement of Theorem A for n odd is the simplest as it does not refer to the Langlands parameter at all. It says in particular that a selfdual irreducible representation of $D^\times$ for a division algebra of index n, an odd integer, must have a non-degenerate invariant symmetric bilinear form. Since $D^\times/k^\times$ is a compact, profinite group, the question is clearly in the realm of finite group theory. However, our proof uses many recent and nontrivial results in the theory of Automorphic representations to achieve this. Recently, Bushnell and Henniart have given a local proof of this case of our result in [BH].

More generally, given a connected reductive algebraic group G/k with an involution θ, and an irreducible θ-selfdual representation π of $G(k)$, we associate, in section 7, an invariant $c_\theta(\pi) \in \{\pm 1\}$. When $\theta = 1$, $c_\theta(\pi)$ gives exactly the information on whether π is orthogonal or symplectic. There is an analogous invariant for Galois representations in chapter 15 of [Rog]. We would like to know if $c_\theta(\pi)$ is the same for two such representations in the same L-packet, and if so, whether $c_\theta(\pi)$ admits a formula in terms of the Langlands parameter. In any case, our proof of Theorem C in section 2 extends to this situation when one has multiplicity one for the relevant global representations. We also show locally that for suitable θ, generic discrete series representations π of a quasi-split G have $c_\theta(\pi) = 1$.

When the dual group ${}^LG$ admits an involution ${}^L\theta$, we define (in section 8) an invariant $c_{{}^L\theta}(\sigma) \in \{\pm 1\}$ for L-homomorphisms σ from W'_k into ${}^LG$ which are equivalent to σ^θ under the connected component of ${}^L G$. When G/k is a quasi-split reductive group with an involution θ stabilizing a Borel subgroup, its maximal torus, and the associated pinned root system, one gets such a dual involution ${}^L\theta$ of ${}^L G$. It will be interesting to make a comparison of $c_{{}^L\theta}(\sigma)$, for σ whose image in ${}^L G$ is not contained in a Levi subgroup, and the $c_\theta(\pi)$ for π, for π in the L-packet associated to σ by the (conjectural) local Langlands correspondence.

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1. **Results on functoriality which we need**

We have the following theorem due to Jiang and Soudry, cf. Theorem 6.4 of [JS1], and Theorem 2.1 of [JS2]. As mentioned in the Introduction, we are thankful to David Soudry for writing one of us ([Sou2]) explaining the completion of their work; see also [Sou1] and [GJR].

Theorem 1.1. *Let G be either a symplectic group or a quasi-split orthogonal group over a non-Archimedean local field k. Let the L-group of G come equipped with its natural representation into $\mathrm{GL}_n(\mathbb{C})$. Then there exists a bijective correspondence between irreducible generic discrete series representations of $G(k)$ and irreducible generic representations of $\mathrm{GL}_n(k)$, with Langlands parameter of the form*

$$\sigma = \sum \sigma_i,$$

where σ_i are pairwise inequivalent, irreducible representations of $W_k' \to \mathrm{GL}_n(k)$ which are all orthogonal except when G is an odd orthogonal group in which case they are all symplectic.

We will also need following theorem due to Vignéras [Vig].

Theorem 1.2. *Let K be a number field, and let G be a quasi-split semi-simple group over K. Let $v_i, i = 1, \cdots, d$ be places of K, and K_{v_i} the corresponding local fields. Suppose that π_i are irreducible generic supercuspidal representations of $G(K_{v_i})$. Then there exists a generic cuspidal automorphic representation Π of $G(\mathbb{A}_K)$ with π_i as the local component of Π at the place v_i, for $i = 1, \cdots, d$.*

The following theorem of Cogdell, Kim, Piatetski-Shapiro, and Shahidi (cf. [CPSS], [CKPSS]) establishes the weak Langlands functoriality from G to GL_n.

Theorem 1.3. *Let G be either a symplectic group or a quasi-split orthogonal group over a number field K. Let $\mathcal{A}^{0,g}(G(\mathbb{A}_K))$ be the set of irreducible generic cuspidal automorphic representations of $G(\mathbb{A}_K)$, and let $\mathcal{A}(\mathrm{GL}_n(\mathbb{A}_K))$ be the set of irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A}_K)$. Then there is a weak functorial lift from $\mathcal{A}^{0,g}(G(\mathbb{A}_K))$ to $\mathcal{A}(\mathrm{GL}_n(\mathbb{A}_K))$.*

The weak Langlands functoriality established by Cogdell, Kim, Piatetski-Shapiro, and Shahidi has been proved to be Langlands functorial at all places of K by Jiang and Soudry, cf. Theorem E of [JS2] when $G = \mathrm{SO}(2m+1)$, and [Sou1], [Sou2] for the remaining cases; a general statement is also in [GJR].

Theorem 1.4. *Let G be either a symplectic group or a quasi-split orthogonal group over a number field K. The weak lift from the set of irreducible generic cuspidal automorphic representations of $G(\mathbb{A}_K)$ to the set of irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A}_K)$ is Langlands functorial at every place of K, i.e., it is the functorial lift at all the places of K.*

An implication for us of these theorems is the following:

Theorem 1.5. *Let K be a number field, and $v_i, i = 1, \cdots, d$ places of K. Let K_{v_i} be the corresponding local fields. Suppose that π_i are irreducible selfdual supercuspidal representations of $\mathrm{GL}_n(K_{v_i})$ whose parameters are either orthogonal, or symplectic for all i. Then there exists a selfdual cuspidal automorphic representation Π on $\mathrm{GL}_n(\mathbb{A}_K)$ with π_i as the local component of Π at the places v_i.*

Proof: Since the parameters of the representations π_i are all orthogonal or are all symplectic, they can be transported, thanks to Theorem 1.1, to generic supercuspidal representations Π_i of $G(K_{v_i})$ for a quasi-split orthogonal or symplectic group G over a number field K. Using Theorem 1.2, these can be globalized into a cuspidal automorphic representation Π on $G(\mathbb{A}_K)$, i.e., with $\Pi_{v_i} \simeq \Pi_i$ for all i. The automorphic representation Π can then be transferred to GL_n by Theorem 1.3, such that the lifted automorphic representation π on $\mathrm{GL}_n(\mathbb{A}_K)$ has the correct local components π_i at all the v_i by Theorem 1.4, proving Theorem 1.5.

□

We note that such a globalization theorem will also follow from a stabilization of Arthur's twisted trace formula for $\mathrm{GL}(n)$. Some partial results are found in [CC], which do not suffice for us, however.

Remark 1.6 It turns out that in our proof of Theorem A, we will also need an analogue of Theorem 1.5 when one of the π_i is a (suitable) generalized Steinberg representation. A soft result, sufficient for our purposes, is achieved in section 3 (see Theorem 3.1) using certain results of Clozel [Clo1, Clo2], Harris [HL], Labesse [Lab], and Moeglin [Mœg] involving the base change from unitary groups to $\mathrm{GL}(n)$.

2. **Positivity of the global sign**

Using the Jacquet-Langlands correspondence (see Theorem 2.5), we can transport a selfdual, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ to a selfdual automorphic representation π^D of $D_K^\times(\mathbb{A})$ of discrete type, where D_K is the central division algebra over K ramified at only two places v_1, v_2 with invariants $\frac{1}{n}$, and $-\frac{1}{n}$. The assertion of Theorem C is that π^D is orthogonal, i.e., that the global sign $c(\pi^D)$ is 1. We will prove this in a more general setup.

Let G be a group, equipped with an involution θ. Call a representation η of G to be θ-selfdual if and only if $\eta^\vee \simeq \eta^\theta$. Here η^θ is defined as $g \mapsto \eta(g^\theta)$, for all g in G. This implies that there exists a G-invariant bilinear form $B' : \eta \times \eta^\theta \to \mathbb{C}$ which by Schur's lemma is unique up to scaling. The bilinear form can also be thought of as a bilinear form $B : \eta \times \eta \to \mathbb{C}$ such that $B(gv, \theta(g)w) = B(v, w)$ for all $g \in G$, $v, w \in \eta$. As θ is of order 2, it is clear that $\tilde{B}$ defined by $\tilde{B}(v, w) = B(w, v)$ also has the same invariance property with respect to G, and thus $\tilde{B} = cB$ for some nonzero $c \in \mathbb{C}$. Since $\tilde{\tilde{B}} = B$, we find that $c^2 = 1$, i.e., $c = \pm 1$. Thus we have associated an invariant $c = c_\theta(\eta) \in \{\pm 1\}$ for any representation η of G with $\eta^\vee \cong \eta^\theta$. The analogue of the invariant $c_\theta(\pi)$ for Galois representations appearing in the context of unitary groups was introduced by Rogawski in chapter 15 of his book [Rog], see lemma 15.1.1, and 15.1.2. When $\theta = 1$, $c_\theta(\pi)$ supplies exactly the information on whether π is orthogonal or symplectic.

Theorem 2.1. *Let G be a reductive algebraic group over a number field K together with an involution θ. Suppose $\Pi = \otimes'_v \Pi_v$ is an irreducible, θ-selfdual automorphic representation of $G(\mathbb{A}_K)$ of central character ω, which occurs discretely in $L^2(G(K)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F), \omega)$ with multiplicity one. Then we have*

$$c_\theta(\Pi) \,=\, \prod_{v \in \mathrm{ram}(G,\Pi)} c_\theta(\Pi_v) \,=\, 1,$$

where ram(G, Π) denotes the finite set of places outside which G_v is quasi-split and Π_v is unramified.

Proof Define a $G(\mathbb{A})$-invariant bilinear form B on Π by

$$(f, g) \longrightarrow \int_{G(K)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f g^\theta d\mu, \qquad \forall\, f, g \in \Pi,$$

where $d\mu$ is an invariant measure on $G(K)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. We check that this is a non-degenerate bilinear form on Π. Note that the space of functions spanned by $\bar{f}$ (the complex conjugate of f), as f varies over Π, gives rise to the representation $\Pi^\vee$, which is isomorphic to Π^θ. Hence by the multiplicity 1 hypothesis, $\bar{f}^\theta \in \Pi$. Taking $g = \bar{f}^\theta$, we see that

$$B(f, g) \,=\, \int f \bar{f} d\mu \,\neq\, 0.$$

Consequently, the bilinear form B on Π is non-degenerate. We have to show that $c_\theta(\Pi) = 1$. (When $\theta = 1$, this is obvious as the bilinear form is evidently symmetric.) It suffices to show that there exist f, g such that $B(f, g)$ and $B(g, f)$ are both positive. But if we take $g = \bar{f}^\theta$ as above, then $B(f, g)$ (resp. $B(g, f)$) is the L^2-norm of f (resp. f^θ). So the θ-sign of Π is positive.

It is left to show that for almost all places v, $c_\theta(\Pi_v) = 1$. This is a consequence of the following result, observed in [PR] for $G = \mathrm{GL}_n$.

Proposition 2.2. *Every irreducible admissible representation π of a quasi-split reductive group H over a local field, satisfying $\pi^\vee \simeq \pi^\theta$ for an involution θ of $H(k)$, has θ-sign 1 whenever π is unramified (for a maximal compact subgroup left invariant by θ). When $H =$GL(n) and $\theta = 1$, the same assertion holds for any π (not necessarily generic, by the theory of degenerate Whittaker models).*

Theorem 2.1 now follows, as does Theorem C.

For our applications, we will need some information at infinity, where we will assume, and this suffices for the purposes of this paper, that the group is $\mathrm{GL}_n(\mathbb{R})$. We note the following lemma.

Lemma 2.3. *Let π be an irreducible $(\mathfrak{g}, K)$-module where $\mathfrak{g}$ is the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ (resp. $\mathrm{GL}_n(\mathbb{C})$), and $K = \mathrm{O}_n(\mathbb{R})$ (resp. U_n) is the maximal compact subgroup of $\mathrm{GL}_n(\mathbb{R})$ (resp. $\mathrm{GL}_n(\mathbb{C})$). Then if π is selfdual, it carries a symmetric bilinear form, i.e., $c(\pi) = 1$.*

Proof: It is a general result due to D. Vogan that any irreducible $(\mathfrak{g}, K)$-module has a minimal K-type which occurs with multiplicity 1. (All that matters for us is that there is a K-type which appears with multiplicity 1.) Since it is well-known that every irreducible, selfdual representation of $\mathrm{O}_n(\mathbb{R})$ (resp. U_n) carries an invariant symmetric bilinear form, the lemma follows. (Note the difference between U_n and SU_n; there are already irreducible symplectic representations of SU_2, but they do not extend to selfdual representations of U_2.)

3. An interlude on globalization

It appears difficult to construct, for general n, selfdual cusp forms on $\mathrm{GL}(n)$ which have a given generalized Steinberg component at one place and a supercuspidal component at another. The following represents a stopgap measure, which happens to be sufficient for our purposes (see section 4).

Theorem 3.1. *Fix $n = 4dm$, with $d, m \geq 1$, and let k be a non-archimedean local field. Let π be a discrete series representation of $GL_n(k)$ with Langlands parameter $\sigma = \sigma_0 \otimes sp_{2m}$, where σ_0 is irreducible and induced by a character χ of the unramified degree $2d$ extension k_{2d} of k such that $\chi^\alpha = \chi^{-1}$, with α denoting the unique element of order 2 of $Gal(k_{2d}/k)$. Then there exists a totally real number field F with $F_v = k$ for a finite place v, and a selfdual, cuspidal automorphic representation Π of $GL_n(\mathbb{A}_F)$, such that (i) $\Pi_v \simeq \pi$ and (ii) Π_u is, for another finite place u of F, a supercuspidal representation π', say, with parameter σ' satisfying $c(\sigma') = c(\sigma)$.*

Proof of Theorem 3.1. Since sp_{2m} is the symmetric $(2m-1)$-th power of the standard representation of $\mathrm{SL}(2, \mathbb{C})$, it is symplectic, and thus

$$c(\sigma) = c(\sigma_0)c(sp_{2m}) = -c(\sigma_0).$$

We begin with a simple lemma.

Lemma 3.2. *Fix $m \geq 1$. Let k' be a non-archimedean local field of residual characteristic p odd, which contains all the $2m$-th roots of unity. Then there exists an irreducible $2m$-dimensional, symplectic representation τ of $W_{k'}$ such that, for any finite unramified extension k''/k', the restriction of τ, to $W_{k''}$ remains irreducible.*

Proof of Lemma. Let E/k' be a ramified cyclic $2m$-extension, which exists by Kummer theory, and write L for the unique cyclic m-extension of k' contained in E. It suffices to construct a character λ of $E^\times$ such that its restriction to $L^\times$ is the quadratic character $\omega_{E/L}$ associated to the quadratic extension E by the class field theory, and such that its restriction to the units $\mathcal{O}_E^\times$ has $2m$ distinct Galois conjugates. For this observe that the exponential mapping induces an isomorphism, which is Galois equivariant, between the additive group $\mathcal{P}_E^r$ and the multiplicative group $1+\mathcal{P}_E^r$ for sufficiently large integer r, and therefore to construct a character on $1+\mathcal{P}_E^r$, it suffices to construct one on the additive group $\mathcal{P}_E^r$. Now observe that by the normal basis theorem, E is a free $k'[G]$-module of rank 1 where G is the Galois group of E over k'. From this it is easy to construct a homomorphism of $\mathcal{O}_{k'}$-modules from $\mathcal{P}_E^r$ to $\mathcal{O}_{k'}$ which is trivial on $\mathcal{P}_L^r$, and has $2m$ distinct Galois conjugates. Composing with an appropriate character of $\mathcal{O}_{k'}$ to $\mathbb{C}^\times$, we obtain a character on $1+\mathcal{P}_E^r$ which has $2m$ Galois conjugates, and which is trivial on $1+\mathcal{P}_L^r$. One can extend this character to $E^\times$ such that its restriction to $L^\times$ is $\omega_{E/L}$, giving us the desired character λ. We set $\tau := \mathrm{Ind}_E^{k'}(\lambda)$.

□

Now we choose a totally real number field F with finite places u, v such that $F_v = k$ and $F_u = k'$. Then we pick a cyclic extension M/F of degree $2d$ in which both v and u are inert (of degree $2d$ over F), such that the unique quadratic extension K of F contained in M is a CM field. Denote the unique prime divisors of v in M, resp. K, by $\tilde{w}$, resp. w, and the corresponding ones over u by $\tilde{w}'$, resp. w'. By construction, $M_{\tilde{w}} = k_{2d}$, $K_w = k_2$, $M_{\tilde{w}'} = k'_{2d}$, and $K_{w'} = k'_2$.

Let η denote the supercuspidal representation of $\mathrm{GL}_{2m}(k')$ associated to τ by the local Langlands conjecture. Fix a conjugate selfdual character χ' of ${k'_{2d}}^\times$ whose restriction to ${k'_d}^\times$ is trivial if and only if the restriction of the given character χ of $k_{2d}^\times$ is trivial on $k_d^\times$. This implies that $c(\sigma_0) = c(\sigma_0')$, where

$$\sigma_0' = \mathrm{Ind}_{k'_{2d}}^{k'}(\chi').$$

We put

$$\sigma' \simeq \sigma_0' \otimes \tau,$$

and note that it is irreducible by the Lemma. Since by construction, $c(sp_{2m}) = c(\tau) = -1$, we also have

$$c(\sigma) = c(\sigma').$$

Now let π' be the supercuspidal representation of $\mathrm{GL}_n(k')$, $n = 4dm$, which is associated to σ' by the local Langlands correspondence. Note that

$$\pi' \simeq I_{k'_{2d}}^{k'}(\chi') \boxtimes \eta,$$

where $\boxtimes$ denotes the functorial product. It is this π' for which Theorem 3.1 holds.

We now need the following two auxiliary results, which are likely known to experts, but we indicate their proofs for completeness.

Theorem 3.3. *Let M/N be a quadratic extension of number fields with non-trivial automorphism α. Denote by S a non-empty finite set of finite places x of M (lying above places y of N) such that the local degrees $[M_x : N_y]$ are all 2. For every x in S, let $\chi_x : M_x^\times \to \mathbb{C}^\times$ be a finite order character such that $\chi_x^\alpha = \chi_x^{-1}$. Then $\{\chi_x \mid x \in S\}$ can be globalized to a α-self-dual character χ of $\mathbb{A}_M^\times/M^\times$ if and only if the restrictions $\chi_x|_{N_y^\times}$, $x \in S$, are all trivial or all non-trivial.*

Theorem 3.4. *Let K be a CM quadratic extension of a totally real number field F, with non-trivial automorphism θ over F. Fix places v, u of F which are inert in K with respective divisors w, w'; put $k = F_v$ and $k' = F_u$. Let St denote the Steinberg representation of $GL_{2m}(K_w)$ and η a selfdual, supercuspidal representation of $GL_{2m}(k')$ whose parameter τ is symplectic. Then there exists a θ-selfdual, cuspidal* *automorphic representation* Π_1 *of* $GL_{2m}(\mathbb{A}_K)$ *such that* $\Pi_{1,w}$ *is* St *and* $\Pi_{1,w'}$ *is* η_2*, the base change of* η *to* $K_{w'} = k'_2$*.*

Claim 3.5: *Theorem 3.3 + Theorem 3.4* $\implies$ *Theorem 3.1*

Proof of Claim 3.5. Preserve the hypotheses of Theorem 3.1, and the constructions in the paragraphs following its statement. Globalize χ and χ' simultaneously, using Theorem 3.3, to an α-selfdual character Ψ of M such that $\Psi_{\tilde{w}} = \chi$ and $\Psi_{\tilde{w}'} = \chi'$. Next apply Theorem 3.3 to deduce the existence of a θ-selfdual, cuspidal automorphic representation Π_1 of $\mathrm{GL}_m(\mathbb{A}_K)$ such that $\Pi_{1,w} \simeq St$ and $\Pi_{1,w'} \simeq \eta_2$. Put

$$\Pi := I_M^F(\Pi_{1,M} \otimes \Psi) \simeq I_K^F(\Pi_1 \boxtimes I_M^K(\Psi)),$$

where I_M^F (resp. I_K^F) denotes the (global) automorphic induction from $\mathrm{GL}(2m)/M$ (resp. $\mathrm{GL}(2dm)/K$) to $\mathrm{GL}(n)/F$ ([AC]); see also [HH]), and the functorial product $\boxtimes$ makes sense here because $I_M^K(\Psi)$ is cyclic monomial. By construction, $\Pi_v \simeq \pi$ and $\Pi_u \simeq \pi'$. Moreover, since π' is supercuspidal, Π must be cuspidal. Finally, since Π_1 and $I_M^K(\Psi)$ are both θ-selfdual, Π is forced to be selfdual.

□

Proof of Theorem 3.3. The condition $\chi^\alpha = \chi^{-1}$ implies that the character χ must be, upon restriction to $\mathbb{A}_N^\times/N^\times$, trivial on the index 2 subgroup consisting of norms from $\mathbb{A}_M^\times/M^\times$, implying that $\chi|_{\mathbb{A}_N^\times}$ is either 1, or is equal to the quadratic character $\omega = \omega_{M/N}$ of $\mathbb{A}_N{}^\times$ associated to the extension M/N. Thus the existence of a global, θ-selfdual character χ implies that $\chi|_{N_y^\times}$ is either the trivial character $\forall x \in S$, or is the non-trivial quadratic character of ω_y (associated to the extension M_x/N_y) $\forall x \in S$.

Conversely, assume that $\chi_x|_{N_y^\times} \equiv 1$ for all $x|y$ in S. Then by Hilbert Theorem 90, there are characters η_x of $M_x^\times$, $x \in S$, such that $\chi_x = \eta_x^\theta \eta^{-1}$. By the Grunwald-Wang theorem, these characters η_x, $x \in S$, can be globalized to a finite order character η of $\mathbb{A}_M^\times/M^\times$. If we put $\chi := \eta^\theta\eta^{-1}$. then it has the given local components χ_x at all the x in S, and moreover, it is θ-selfdual.

Assume next that $\chi_x|_{N_y^\times} = \omega_y$ for all $x|y$ in S. Let $\tilde{\omega}$ be a character of $\mathbb{A}_M^\times/M^\times$ which restricts to ω on $\mathbb{A}_N^\times/N^\times$. For every x in S, put $\psi_x = \chi_x\tilde{\omega}_x^{-1}$, which restricts to the trivial character on $N_y^\times$ for all $x \in S$. So, by the previous case, there is a θ-selfdual character Ψ of $\mathbb{A}_M^\times/M^\times$ with components ψ_x at $x \in S$. We are now done by setting $\chi = \Psi\tilde{\omega}^{-1}$.

□

Proof of Theorem 3.4. Preserving the hypotheses of the Theorem, denote by $\tilde{\omega}$ an idele class character of the CM field K whose restriction to the idele class group of F is the quadratic character $\omega = \omega_{K/F}$ attached to K/F; then $\tilde{\omega}$ is necessarily conjugate selfdual. Fix a finite set S of finite places of K which are of degree 1 over F with $|S| \geq 2$. Since η is a supercuspidal representation of $\mathrm{GL}_{2m}(F_u)$ of symplectic type, we may, by using Theorem 1.5, globalize it to be the u-component of a selfdual, cuspidal automorphic representation Π_0 of $\mathrm{GL}_{2m}(\mathbb{A}_F)$, such that the set of places where Π_0 is supercuspidal includes, besides u, places of F below S. Then the base change $\Pi_{0,K}$ to $\mathrm{GL}(2m)/K$ is still cuspidal as it has supercuspidal components, and moreover, it is θ-selfdual.

Let G denote an anisotropic unitary group over F associated to K/F such that G_v and G_w are quasi-split; it may be defined either by a hermitian form in $2m$ variables over K or by a division algebra $\mathbb{D}/K$ of dimension m^2, ramified only at a subset of S and split at all the other places (including w, w'), and equipped with an involution $\tilde{\theta}$ which restricts to θ on K. Then by [Clo2, Lab] or [HL], Theorem 2.4.1, we know that either $\Pi_{0,K}$ or $\Pi_{0,K} \otimes \tilde{\omega}$ descends to a cuspidal automorphic representation π_0^G of $G(\mathbb{A}_F)$, which is functorial almost everywhere and is compatible with local Langlands transfer where it is available. Since $\Pi_{0,w'} \simeq \eta_2$ is a discrete series, we know by Moeglin [Mœg], cf. sections 6-8, that the local descent $\beta := \pi_{0,u}^G$ belongs to a stable discrete series L-packet of $G(F_u)$ attached to the parameter $\psi : W'_{k'} \to {}^LG$ defined by $\Pi_{0,w'}$. (In [Mœg], she considers a larger class of "θ-discrete" representations, and associates canonically an elliptic, stable L-packet on a unique twisted endoscopic group of the form $U(n_1) \times U(n_2)$, but in the discrete series case either n_2 or n_1 is zero, and which case occurs depends on whether the representation itself, or its $\tilde{\omega}_{w'}$-twist, descends.) Since $\Pi_{0,w'}$ is conjugate selfdual, its parameter defines an admissible homomorphism $\psi_2 : W'_{k'_2} \to {}^LG$ whose image is contained in no proper Levi, and either ψ_2 or $\psi_2 \otimes \tilde{\omega}_{w'}$, but not both, extends to give the parameter ψ on $W'_{k'}$. Let us write $\mu = 1$ or $\tilde{\omega}$, so that $\Pi_{0,K} \otimes \mu$ descends to π_0^G. By Lemma 15.1.2 of [Rog], $\mu_{w'}$ is trivial on ${k'}^\times$ if and only if the θ-sign of its parameter, namely $c_\theta(\tau_2)$, with τ_2 denoting $\tau_{|_{k'}}$, is 1. There is also a weak base change (see [Clo2] or [HL], Theorem 2.2.2), transferring π_0^G back to a cusp form Π' on $\mathrm{GL}_{2m}(\mathbb{A}_K)$, which is conjugate selfdual and is functorial at almost all places. (We use Jacquet-Langlands in addition to base change if G is defined by $\mathbb{D}$.) Then by the strong multiplicity one theorem for $\mathrm{GL}(2m)/K$, Π' and $\Pi_{0,K} \otimes \mu$ must be isomorphic, and in particular, $\Pi'_{w'}$ is just the supercuspidal representation $\eta_2 \otimes \mu_{w'}$.

Next we take up the problem of globalizing η_2 and St simultaneously to a conjugate selfdual cusp form Π_1 on $\mathrm{GL}(2m)/K$. We construct $\Pi_1 \otimes \mu$ as the base change of a suitable cusp form π^G on G/F. Note that by hypothesis, $c(\tau) = c(sp_{2m}) = -1$, which implies that $c_\theta(\tau_2) = c_\theta(sp_{2m})$, and so we have no obstruction, and $St \otimes \mu_w$ will descend to $G(F_v)$.

Proposition 3.6 *There exists a cuspidal automorphic representation π^G of $G(\mathbb{A}_F)$ such that its components at v, u and the archimedean places are sufficiently regular discrete series representations, and moreover, up to modifying μ by a character which restricts to the trivial character of $\mathbb{A}_F^\times/F^\times$, the local component π_v^G, resp. π_u^G, base changes to $St \otimes \mu_w$, resp. $\eta_2 \otimes \mu_{w'}$, on* $\mathrm{GL}(2m)/K_w$*, resp. on* $\mathrm{GL}(2m)/K_{w'}$.

Proof of Proposition 3.6. Let St^G denote the Steinberg representation of $G(F_v)$. By appealing to a result of Clozel on limit multiplicities (cf. Theorem 2 of [Clo1]), we get the existence of a cusp form π^G with sufficiently regular discrete series (resp. supercuspidal) components at infinity (resp. S), such that $\pi_v^G \simeq St^G$ and $\pi_u^G \simeq \beta \in \Pi_u(\psi)$. Denote by Π_1 the base change of π^G to GL_{2m}/K. As we have seen above, the local base change at u sends β to $\eta_2 \otimes \mu_{w'}$. Moreover, one sees (cf. section 3.9 of [Lab]) that the local base change $\Pi_{1,w}$ of St^G to $\mathrm{GL}_{2m}(K_w$ is an unramified twist of St. Indeed, the global base change is achieved by comparing the trace of functions f on $G(\mathbb{A}_F)$ with the twisted trace of functions φ on $\mathbb{D}^\times(\mathbb{A}_K)$, and locally at v, Proposition 3.9.2 of *loc. cit.* matches the Kottwitz functions f_v on $G(F_v)$ and φ_w on $\mathrm{GL}_{2m}(K_w)$, and these functions are stabilizing and have zero traces on all the irreducible unitary representations except for those occurring as subquotients, up to unramified twists, of principal series representations admitting the trivial representation as the Langlands quotient. Note that unramified twists of non-Steinberg representations of $\mathrm{GL}_{2m}(K_{w'})$ occurring as such subquotients are non-generic and cannot occur as local components of cusp forms on $\mathrm{GL}(2m)/K$. It follows that $\Pi_{1,w} \simeq St \otimes \nu$ for an unramified character ν, which must differ from μ_w by a character which is trivial on $F_v^\times$.

□

To conclude, we get a θ-selfdual, cuspidal automorphic representation Π_1 of $\mathrm{GL}_m(\mathbb{A}_K)$ associated to π^D, which, when twisted by μ^{-1}, will satisfy all the assertions of Theorem 3.4.

□

4. **Sign in the Level 1 case**

In this section we prove Theorem A about irreducible selfdual representations of $D^\times/D^\times(1)$. The global proofs in this paper depend most crucially on this local input, as it represents the simplest of the situations for Theorem A.

Various aspects of the representation theory of $D^\times/D^\times(1)$ are analyzed in the work of Silberger and Zink in [SZ]. We begin with some notation, and recalling the parametrization of the irreducible representations of $D^\times/D^\times(1)$ from [SZ], where they unfortunately call them level zero representations.

As in the rest of the paper, let D be a division algebra with center a non-Archimedean local field k, and of index n. Let $n = ef$, and let k_f be the unramified extension of k of degree f, contained in D. Let D_f be the centralizer of k_f in D which is a division algebra with center k_f and of index e. A character χ of $k_f^\times$ will be called regular if all its Galois conjugates are distinct. For a character χ of $k_f^\times$, let $\tilde{\chi}$ be the character of $D_f^\times$ obtained by composing with the reduced norm mapping $\mathrm{Nrd} : D_f^\times \to k_f^\times$. If the character χ is tame, i.e., trivial on $k_f^\times(1)$, then the character $\tilde{\chi}$ of $D_f^\times$ can be extended to a character of $D^\times(1)D_f^\times$ by declaring it to be trivial on $D^\times(1)$, which, by abuse of notation, we again denote by $\tilde{\chi}$.

With this notation, it follows from Clifford theory as in [SZ] that irreducible representations of $D^\times/D^\times(1)$ have dimensions f which are divisors of n, and that there is a bijection between irreducible representations of $D^\times/D^\times(1)$ of dimension f and regular characters of $k_f^\times$ which are trivial on $k_f^\times(1)$ (modulo the action of the Galois group of k_f over k on the set of such characters) obtained by inducing the character $\tilde{\chi}$ of $D^\times(1)D_f^\times$ to $D^\times$.

To analyze whether these representations, when selfdual, are orthogonal or symplectic, we state the following general proposition, whose straightforward proof will be omitted.

Proposition 4.1. *Let H be a normal subgroup of a group G of index $f > 1$, such that G/H is a cyclic group of order f. Let ϖ be an element of G whose image in G/H is a generator of the cyclic group G/H. Let π be an irreducible representation of G of dimension f whose restriction to H contains a character $\chi : H \to \mathbb{C}^\times$, so that $\pi = \mathrm{Ind}_H^G \chi$. Then,*

(1) If π is selfdual, f is even, say $f = 2d$, which we assume is the case in the rest of the proposition.

(2) The representation π is selfdual if and only if $\chi^{-1} = \chi^{<d>}$ where $\chi^{<d>}(h) = \chi(\varpi^d h \varpi^{-d})$ for $h \in H$.
(3) ϖ^f acts on π by a scalar $C(\pi)$ on the representation space π. If π is selfdual, $C(\pi) = \pm 1$, and $C(\pi) = 1$ if and only if π is an orthogonal representation.
(4) If π is selfdual, it is orthogonal if and only if $\det \pi(\varpi) = -1$.

Continuing now with the representations of $D^\times$ of dimension f with ef, let ϖ be an element of $D^\times$ which normalizes k_n, an unramified extension of k inside D of degree n over k, such that $\varpi^n = \varpi_k$, a uniformizer in k.The element ϖ of $D^\times$ projects to a generator of the cyclic group $D^\times/D^\times(1)D_f^\times \cong \mathbb{Z}/f\mathbb{Z}$, and as ϖ^f centralizes k_f, it lies in D_f. Since $(\varpi^f)^e = \varpi_k$, it follows that the reduced norm of ϖ^f is ϖ_k. From the previous proposition, we conclude the following corollary.

Corollary 4.2. *Let χ be a regular character of $k_f^\times$, and π_χ the associated representation of $D^\times$ of dimension f. Then π_χ is selfdual if and only if f is even, say $f = 2d$, and the character χ restricted to $k_d^\times$ is trivial on the index 2 subgroup consisting of norms from $k_f^\times$. Assuming π_χ to be selfdual, it is orthogonal if and only if χ restricted to $k^\times$ is trivial.*

Recalling that the Weil group $W_{k_f/k}$ sits in the exact sequence,

$$1 \to k_f^\times \to W_{k_f/k} \to \mathrm{Gal}(k_f/k) \to 1,$$

and there is again an element ϖ in $W_{k_f/k}$ which goes to the generator of the Galois group of k_f over k, and whose f-th power is a uniformizer in k_f, we have the following corollary for representations of the Weil group.

Corollary 4.3. *For a regular character μ of $k_f^\times$, let σ_μ be the induced representation of $W_{k_f/k}$ of dimension f. Then σ_μ is selfdual if and only if f is even, say $f = 2d$, and the character μ restricted to $k_d^\times$ is trivial on the index 2 subgroup consisting of norms from $k_f^\times$. Assuming σ_μ to be selfdual, it is orthogonal if and only if μ restricted to $k^\times$ is trivial, or if and only if $\det \sigma_\mu$ is nontrivial.*

We will give a proof of Theorem A below without knowing the precise Langlands parameter attached to π_χ. In any case, we believe the following question has an affirmative answer:

Question 4.4 Does the Langlands parameter of the representation π_χ of dimension f of $D^\times$ equal $\sigma_\mu \otimes sp_e$, with σ_μ the f-dimensional representation of W_k induced by the character

$$\mu := \chi \omega_2^{e(f-1)} : k_f^\times \to \mathbb{C}^\times,$$

where ω_2 is the quadratic unramified character of $k_f^\times$?

Remark 4.5 : We have not been able to find a precise reference dealing with this question. However, we should point out that in [SZ] Silberger and Zink suggest without proof, see the Remark on page 182 of *loc. cit.*, that the Langlands parameter of π_χ is $\sigma_\mu \otimes sp_e$, with $\mu = \chi \omega_2^{f-1}$. There is no discrepancy with our formula above unless e and f are both even. Results of 4.6 below show, however, that their formula cannot be correct for all e. Finally, Henniart has informed us that he is working on a paper with Bushnell to answer Question 4.4.

4.6 Proof of Theorem A for level 1 representations. Let π be an irreducible representation of $D^\times$ of level 1 as above. The parameter of π is necessarily of the form

$$\sigma = \sigma_\nu \otimes sp_e, \quad \text{with} \quad \sigma_\nu = \mathrm{Ind}_{W_{k_{2d}}}^{W_k}(\nu) \otimes sp_e,$$

for a tame character ν of $W_{k_{2d}}$ satisfying $\nu^\theta = \nu^{-1}$, where θ denotes the non-trivial automorphism of k_{2d}/k_d. The irreducibility of σ also forces ν to be distinct from ν^ξ for any ξ in $\mathrm{Gal}(k_{2d}/k)$.

First suppose e is 1. Since the central character of π_χ is associated, by the local correspondence, to the determinant of σ_ν, the former is trivial if and only if the latter is also trivial. The Corollaries 3.2 and 3.3 then imply that π_χ is orthogonal if and only if σ_ν is symplectic. This argument works for e *odd* in the same way.

We may now assume that $e = 2m$ and $f = 2d$ are both *even*, so that $n = 4dm$. Thanks to Theorem 3.1, we can find a totally real number field F with $F_v = k$ at a place v, and a cuspidal, selfdual automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ with Π_v in the discrete series corresponding to π, and at another finite place u, $\pi' := \Pi_u$ is supercuspidal. Let $\mathbb{B}/F$ be a division algebra of index n over F which ramifies only at u, v such that $\mathbb{B}_v = D$ and $\mathbb{B}_u = D'$, where D' is also a division algebra. Applying Theorem 2.1 to the automorphic representation $\Pi^{\mathbb{B}}$, say, of $\mathbb{B}(\mathbb{A}_{F_0}^\times)$ associated to Π, we conclude, since $\beta_{v_0} \simeq \pi$, that $c(\pi) = c(\pi')$, where π' corresponds to the supercuspidal representation Π_u. By using Theorem 1.5, we can also find a cuspidal, selfdual automorphic representation Π_0 of $\mathrm{GL}_{2m}(\mathbb{A}_F)$ such that $\Pi_{0,u} = \Pi_u$ and $\pi'' := \Pi_{0,v}$ supercuspidal, with the parameters of these two supercuspidals being both symplectic or both orthogonal. In fact, we have freedom in choosing π'', and we may arrange it to correspond to a level 1 representation of $D^\times$. Applying Theorem 2.1 again, we get $c(\pi'') = c(\pi')$, and since the parameter σ'' of π'' is trivial on $\mathrm{SL}(2, \mathbb{C})$, we know by what we did above that $c(\pi'') = -c(\sigma'')$. It follows that $c(\pi)$ equals $-c(\sigma)$ as desired.

□

Remark 4.7: To be able to make use of representations of level 1, we must know that there is an irreducible selfdual representation of $D^\times/D^\times(1)$ of dimension f for any *even* divisor $f = 2d$ of n. This reduces to a question over finite fields. A character of $\mathbb{F}_{q^{2d}}$ is regular if and only if it does not factor through the norm mapping of an intermediate field, say $\mathbb{F}_{q^s}$, in which case it has an order divisible by $(q^s - 1)$, whereas a character of $\mathbb{F}_{q^{2d}}$ gives rise to a selfdual representation if and only if it arises from the circle group of norm 1 elements of $\mathbb{F}_{q^{2d}}$, denoted $\mathbb{S}^1(\mathbb{F}_{q^d})$, through the map $x \to x/\bar{x}$ for $x \in \mathbb{F}_{q^{2d}}$. Since $\mathbb{S}^1(\mathbb{F}_{q^d})$ is a cyclic group of order $(q^d + 1)$, one can consider characters on it of order $(q^d + 1)$, which will then not arise from an intermediate field through the norm mapping.

5. Proof of Theorem A for $D^\times$

Let π' be an irreducible, selfdual representation of $D^\times$ of parameter σ, and associated discrete series representation π of $\mathrm{GL}_n(k)$. First assume that π is supercuspidal.

When n is *odd*, embed π, by using Theorem 1.5, as a local component Π_v of a selfdual, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_K)$, for a number field K with $K_v = k$. Let E/K be a cyclic extension of degree n such that v splits completely in E, and denote by Π_E the base change of Π to GL_n/E as defined by Arthur and Clozel ([AC]). Let $\mathbb{B}$ be the division algebra of index n over E, ramified only at the places of E above v, such that $\mathbb{B}_v := \mathbb{B} \otimes_K K_v \simeq D^n$. (By class field theory, such a division algebra exists.) Let $\Pi^{\mathbb{B}}$ be the automorphic representation of $\mathbb{B}^\times(\mathbb{A}_E)$ associated to Π_E by the Jacquet-Langlands correspondence ([Bad]), which appears with multiplicity one in the space of automorphic forms on $\mathbb{B}^\times/E$. Then the component of $\Pi^{\mathbb{B}}$ at $E_v := E \otimes_K K_v \simeq K_v^n$ is necessarily isomorphic to $\pi'^{\otimes n}$. Since at any place w of E not lying over v, $\mathbb{B}_w^\times$ is by construction $\mathrm{GL}_n(E_w)$, we see by applying Theorem 2.1 that

$$c(\pi')^n \,=\, c(\pi'^{\otimes n}) \,=\, c(\Pi^{\mathbb{B}}) \,=\, 1.$$

It follows, by the oddness of n, that $c(\pi') = 1$, proving Theorem A in this case.

Now suppose n is even, still with π being supercuspidal. Now globalize π, again using Theorem 1.5, to a selfdual, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_K)$, with $K_v = k$ and $\Pi_v = \pi$. Moreover we can arrange this Π in such a way that, at a second finite place u, the local component Π_u is a selfdual, supercuspidal representation of level 1, i.e., corresponding to a representation of *level* 1 of a division algebra over K_u of index n. Now choose a global division algebra $\mathbb{D}$ of index n over K such that $\mathbb{D}$ is ramified only at u, v, with $\mathbb{D}_v \simeq D$, and denote by $\Pi^{\mathbb{D}}$ the automorphic representation of $\mathbb{D}^\times(\mathbb{A}_K)$ associated to Π. Applying Theorem 2.1, we obtain

$$c(\pi')c(\Pi_u^{\mathbb{D}}) \,=\, c(\Pi^{\mathbb{D}}) \,=\, 1.$$

In section 4, we proved Theorem A in the level 1 situation. Thus we get

$$c(\pi') \,=\, c(\Pi_u^{\mathbb{D}}) \,=\, -c(\sigma_u) \,=\, -c(\sigma),$$

as asserted.

We have now achieved a proof of Theorem A for those representations π' of $D^\times$ for which the corresponding representation π of $\mathrm{GL}_n(k)$ is supercuspidal, thus the Langlands parameter is trivial on the $\mathrm{SL}(2,\mathbb{C})$ part of W_k'. This restriction has been imposed on us as we do not have the globalization theorem (Theorem 1.5) available for general discrete series representations.

Let π be a non-supercuspidal discrete series representation of $\mathrm{GL}_n(k)$. Then its parameter σ must be of the form $\tau \otimes sp_b$, $n = ab$, with sp_b being the unique b-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ and τ an irreducible selfdual a-dimensional representation of W_k.

Let Σ be a selfdual cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A}_K)$ whose local component at the place v of K with completion k has Langlands parameter τ. We may assume, thanks to Theorem 1.5, that at some other finite place, say u, Σ_u is supercuspidal of level 1, whose parameter τ_u is of the same parity as τ. We may take τ_u of the form $\mathrm{Ind}_{L_a^\times}^{W_k}(\chi_1)$, where χ_1 is a certain character of $L_a^\times$ trivial on $L_a^\times(1)$ where L_a is the unique unramified extension of K_u of degree a. By the work of Moeglin and Waldspurger ([MW]), Σ gives rise to a selfdual representation in the residual spectrum of $\mathrm{GL}_n(\mathbb{A}_K)$ denoted by $\Sigma[b]$. Next, by the global Jacquet-Langlands correspondence, the representation $\Sigma[b]$ of $\mathrm{GL}_n(\mathbb{A}_K)$ can be transported to an automorphic representation $\Sigma'[b]$ of $\mathbb{D}^\times(\mathbb{A}_K)$ (of multiplicity one), where $\mathbb{D}$ is a division algebra of index n over K, which is ramified only at u, v, with $\mathbb{D}_v = D$. Again, by Theorem 2.1, the corresponding representation of $\Sigma'[b]$ is orthogonal at v if and only if it is so at u. Since by construction, $\pi' \simeq \Sigma'[b]_v$, we get

$c(\pi') = c(\Sigma'[b]_u)$. Put $B := \mathbb{D}_u$ and $\pi_1 = \Sigma'[b]_u$. The proof is completed by noting that an irreducible representation π of $D^\times/D^\times(1)$ of dimension f dividing n with $e = n/f$ has Langlands parameter which is $\mathrm{Ind}_{k_f^\times}^{W_k}(\chi) \otimes sp_e$ where χ is a certain character of $k_f^\times$, and that Theorem A is true for such representations of $D^\times$.

The proof of Theorem A is now complete for $D^\times$.

□

6. Proof of Theorem A for selfdual representations on $\mathrm{GL}_m(D)$

In this section we extend the results above for $D^\times$ to the case of selfdual irreducible discrete series representations π' of $\mathrm{GL}_m(D)$, with $m > 1$. Put $n = md$, where d is the index of D. We need to prove the following:

Theorem 6.1. *An irreducible, selfdual, discrete series representation π' of $\mathrm{GL}_m(D)$, where D is a division algebra over a non-archimedean local field k of index d, is orthogonal if d is odd. If d is even, and m is odd, then the representation is orthogonal if and only if its parameter is symplectic. If both m and d are even, then the representation is orthogonal.*

Proof. Assume first that the corresponding representation π of $\mathrm{GL}_n(k)$ is supercuspidal.

Suppose d is odd. As π is selfdual and supercuspidal, we may globalize it (by applying Theorem 1.5) to a selfdual, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_K)$ with π as its local component at $K_v = k$. Let E be a cyclic extension of K of degree d such that v splits into d places. Let Π_E denote the base change of Π to E. Let $\mathbb{B}$ be a central division algebra over E of dimension d^2 over E such that $\mathbb{B} \otimes_K K_v \cong D^n$, and such that $\mathbb{B}$ has no other ramification. That there is such a division algebra $\mathbb{B}$ follows from class field theory.

Let $\Pi_E^{\mathbb{B}}$ denote the automorphic representation of $\mathrm{GL}_m(\mathbb{B}(\mathbb{A}_E))$ obtained from Π_E by the Jacquet-Langlands correspondence ([Bad]). The component of Π_E^D at $E_v := E \otimes_K K_v \simeq k^d$ is isomorphic to ${\pi'}^{\otimes d}$. At every place u of E not lying over v, the u-component of $\Pi_E^{\mathbb{B}}$ is a representation of $\mathrm{GL}_n(E_u)$. Applying Theorem 2.1, we find that the d-th power of $c(\pi')$ is trivial, and therefore π' is an orthogonal representation (as d is odd).

Now let the index of D be $d = 2r$. In this case there is a division algebra $\mathbb{D}$ over a number field K of index $2mr$ such that $\mathbb{D}$ gives rise to D at one place v, say, necessarily with index, say $\frac{m}{2mr} \in \mathbb{Q}/\mathbb{Z}$ there, and with indices $\frac{-1}{2mr} \in \mathbb{Q}/\mathbb{Z}$ at m other places, call them $u_1, \ldots, u_m$. We may further take $\mathbb{D}$ to be split at all the remaining places. The existence of such a division algebra follows from class field theory. This global division algebra becomes split, or totally ramified, at every place other than v. Since one understands the parity at the split and totally ramified places, the parity at the unique remaining place v follows, thanks to the global sign being 1 by Theorem 2.1. We are now done when π is supercuspidal.

Now assume that π is not supercuspidal, with its parameter being of the form $\sigma = \tau \otimes sp_b$, $n = ab$, with τ an irreducible, selfdual a-dimensional representation of W_k. The proof now proceeds as in the last paragraph of section 5 except for the additional subtlety that the global Jacquet-Langlands correspondence might produce a Speh-like representation on $\mathrm{GL}_n(D)$ whereas we want to make conclusions about discrete series representations of $\mathrm{GL}_n(D)$. To deal with this possibility, we note the following proposition, which, when combined with the fact that the Aubert-Zelevinsky involution interchanges Speh-modules with the generalized Steinberg representations, completes the proof of Theorem A. This involution $\pi \mapsto i(\pi)$ (cf. [Bad], section 2.6, for example) is defined on the Grothendieck group of smooth representations of a p-adic reductive group G as an alternating sum of parabolically induced representations of the various Jacquet modules of π, and sends an irreducible to another irreducible $|i(\pi)|$ up to sign.

Proposition 6.2. *Let G be a reductive algebraic group over a non-archimedean local field k. Then an irreducible representation π is orthogonal if and only if the corresponding irreducible representation $|i(\pi)|$, associated by the Aubert-Zelevinsky involution i, is orthogonal.*

The proof of this proposition follows from the following lemma using the fact that both induction and Jacquet functor take real representations to real representations.

Lemma 6.3. *An irreducible unitary representation (π, V) of a p-adic group G carries a nonzero symmetric bilinear form $B : V \times V \to \mathbb{C}$ if and only if π is defined over $\mathbb{R}$, i.e., there is a G-invariant real subspace W of V such that $V = W \otimes_{\mathbb{R}} \mathbb{C}$.*

This lemma is well-known for finite groups, and most of the proofs in the literature work in this infinite dimensional context too (assuming only that Schur's lemma holds, which of course it does for p-adic groups).

7. Questions on θ-selfdual representations

Let G be a group, and θ an automorphism of G with $\theta^2 = 1$. Let π be an irreducible representation of G. In this section the group G will be any one of the finite, real, p-adic, or adelic group, and the representations, as well as their contragredients, will be understood in the corresponding category. Recall that π is called θ-selfdual if $\pi^\vee \cong \pi^\theta$. In section 2, we have associated an invariant $c_\theta(\pi) \in \{\pm 1\}$ for any representation π of G with $\pi^\vee \cong \pi^\theta$, which for $\theta = 1$ furnishes the information on whether π is orthogonal or symplectic.

Example : Let E be a quadratic extension of a local field F with σ as the Galois automorphism of E over F. For any algebraic group G over F, this gives an involution, call it θ, on $G(E)$, and thus we have associated an invariant $c_\theta \in \{\pm 1\}$ for any representation π of $G(E)$ with $\pi^\vee \cong \pi^\theta$.

Remark: If G is an algebraic group over a finite field $\mathbb{F}$, $\mathbb{E}$ a quadratic extension of $\mathbb{F}$, and θ the automorphism of order 2 on $G(\mathbb{E})$ induced by the Galois action on $\mathbb{E}$, then the invariant c_θ has been studied in [Pra1]. It follows from the results there that for irreducible representations π of $G(\mathbb{E})$ with $\pi^\vee \cong \pi^\theta$, the invariant $c_\theta(\pi) = 1$ for those irreducible representations π which are linear combinations of Deligne-Lusztig representations.

Question 7.1. (a): *Let G be a reductive p-adic group, and π_1 and π_2 two irreducible tempered representations of G. Is it true that if $\pi_1^\vee \cong \pi_1^\theta, \pi_2^\vee \cong \pi_2^\theta$, and π_1 and π_2 are in the same L-packet, then $c_\theta(\pi_1) = c_\theta(\pi_2)$?*

(b): *If part (a) is true, is there an expression for $c_\theta(\pi)$ in terms of the Langlands parameter of π? The results in this paper suggest that $c_\theta(\pi)$ depends not only on* *the L-group, but also the inner form (as the result is different for* GL(n) *and for division algebras), and that it varies as one varies the inner forms as in Kottwitz's paper* [Kot].

(c): *If π is automorphic with $\pi^\vee \cong \pi^\theta$, is $c_\theta(\pi) = 1$ (like it is for π occurring discretely with multiplicity one; cf. Theorem 2.1)?*

When G is SL(n), the elements of any L-packet of representations are permuted transitively by conjugation by elements of GL(n), and therefore, at least in this case, whether a representation of SL(n) is orthogonal or symplectic is a property of the L-packet.

Here is a local result concerning generic, square-integrable, θ-selfdual representations in the style of Theorem 2.1.

Proposition 7.2. *Let G be a quasi-split reductive group over a local field, with maximal unipotent subgroup N, equipped with an involution θ which preserves N. Let ψ be a non-degenerate character of N satisfying $\psi^\theta = \overline{\psi}$. Then for any ψ-generic discrete series representation π of G which is θ-selfdual, we have $c_\theta(\pi) = 1$.*

Proof, For any tempered ψ-generic π, one has an infinitesimal embedding $\xi \mapsto W_\xi$, of π into $L^2(N\backslash G, \psi)$, and the square-integrability of π implies, by a theorem of Harish Chandra, that π occurs discretely; denote by $\mathcal{W}_\pi$ the image of π. Moreover, one knows that $\mathcal{W}_\pi$ has multiplicity one. As $\psi^\theta = \overline{\psi}$ by hypothesis, we may define a bilinear form B on π by

$$(\xi, \xi') \ \to \ \int_{N\backslash G} W_\xi(g) W_{\xi'}^\theta(g) dg.$$

Since $\pi^\theta \simeq \pi^\vee$, the function $\overline{W}_\xi^\theta$ lies, by multiplicity one, in $\mathcal{W}_\pi$. So we may take W'_ξ to be $\overline{W}_\xi^\theta$ and find that $B(\xi, \xi')$ is the L^2-norm of W_ξ. The argument is now completed as in the proof of Theorem 2.1.

7.3 Dual involutions and dual signs

Now suppose that G is a connected reductive group over a local field k, with the Langlands dual group ${}^LG$ admitting a an involution ${}^L\theta$. Such an involution on ${}^LG$ certainly exists when G is quasi-split admitting an involution θ preserving a Borel subgroup B, a maximal torus $T \subset B$, and the pinned root system Ψ.

Consider any L-parameter

$$\sigma : W_k \ \to \ {}^LG,$$

not lying in a Levi subgroup of ${}^LG$. Suppose σ is conjugate to $\sigma^{{}^L\theta}$ by an element t of the connected component $G^\vee$ of ${}^LG$. Then we can associate a "sign" $c_{{}^L\theta}(\sigma)$ as follows: It is easy to see that tt^θ must belong to the center ${}^LZ$ of ${}^LG$. This implies in particular that t and t^θ commute, and that tt^θ is a θ-invariant element of ${}^LZ$. The element t is well defined modulo ${}^LZ$, and hence tt^θ is well-defined in the Tate cohomolgy group $\hat{H}^0(C, {}^LZ) = {}^LZ^C/N$, where C is the cyclic group $\{1, \theta\}$, ${}^LZ^C$ is the group of C-fixed points of ${}^LZ$, and N the subgroup of norms. We will write $c_{{}^L\theta}(\sigma)$ for the class of tt^θ. If σ is the parameter of π, it will be interesting to compare $c_\theta(\pi)$ with $c_{{}^L\theta}(\sigma)$. When G is GL$(n)/k$ with $\theta(g) = {}^tg^{-1}$, we have $Z^C = \{\pm 1\}$ and $N = \{1\}$, so $c_{{}^L\theta}(\sigma)$ takes values in $\{\pm 1\}$.

8. **Rationality**

The question about selfdual representations being orthogonal or symplectic is part of the more general question about *field of definition* of a representation. For example, by (the well known) lemma 6.3, a selfdual, unitary representation is orthogonal if and only if it is defined over $\mathbb{R}$.

Let G be a group, and π an irreducible representation of G over $\mathbb{C}$. Put

$$\mathcal{G}_\pi = \{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) | \pi^\sigma \cong \pi\}.$$

If either G is finite, or G is a reductive p-adic group, and π is supercuspidal with finite order central character, then $\mathcal{G}_\pi$ is known to be a subgroup of finite index of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, and this defines a finite extension K of $\mathbb{Q}$. Call K the field of definition of π. (If π is finite dimensional, it is the field generated by the character values of π.)

Assume that π is finite dimensional. Then associated to π, there is a division algebra $\mathcal{D}_\pi$ with center K which measures the obstruction to π being defined over K, called Schur Algebra, thus $\mathcal{D}_\pi = K$ if and only if π can be defined over K. Let the dimension of $\mathcal{D}_\pi$ over K be d^2; the integer d is called the Schur index of π.

Now let $\sigma \to \pi_\sigma$ be the local Langlands correspondence between irreducible representations of $\mathrm{Gal}(\bar{k}/k)$ of dimension n, and representations of $\mathrm{GL}_m(D)$ where D is a division algebra of index r with $rm = n$. Here and in what follows, we normalize the Langlands correspondence by multiplying by the character $x \to |x|^{(n-1)/2}$ where $x \in k^\times$. This normalized Langlands correspondence, which does not affect rationality (reality!) questions at infinity, is what is Galois equivariant on the coefficients; see for example, Henniart [Hen2].

Question: How are $\mathcal{D}_\sigma$ and $\mathcal{D}_{\pi_\sigma}$ related? This time the answer a priori might depend not just on the index of the division algebra, but on its class in the Brauer group. However, we propose the following conjecture, suggesting in particular that this is not the case.

Conjecture 8.1. *Let D be a division algebra of index r over a non-archimedean local field k, π an irreducible representation of $\mathrm{GL}_m(D)$, and σ_π the associated n-dimensional representation of the Weil-Deligne group of k for $n = mr$. Let $\mathcal{D}_\sigma$ and $\mathcal{D}_{\pi_\sigma}$ be the associated Schur algebras with center a number field K (which is a cyclotomic field). Then the following happens:*

1. *If π is not selfdual, or π is selfdual with $c(\sigma) = c(\pi_\sigma)$, then $\mathcal{D}_\sigma = \mathcal{D}_{\pi_\sigma}$.*
2. *If π is selfdual and $c(\pi) = -c(\pi_\sigma)$, then the answer depends on the degree K over $\mathbb{Q}$ which is a certain totally real extension of $\mathbb{Q}$. If $[K : \mathbb{Q}]$ is even, then the invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_{\pi_\sigma}$ are the same except at infinite places, where the invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_{\pi_\sigma}$ differ by $1/2$. If $[K : \mathbb{Q}]$ is odd, then in particular there are odd number of places in K over p. The invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_{\pi_\sigma}$ are the same except at infinite places, and at the places in K above p, where the invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_{\pi_\sigma}$ differ by $1/2$.*

One case of the conjecture is especially simple to state. This is when r, the index of D is odd, and π is a selfdual representation of $\mathrm{GL}_m(D)$ for m odd. In this case, irreducible selfdual representations of the Galois group of dimension n, or irreducible selfdual representation of $\mathrm{GL}_m(D)$ exist only in even residue characteristic, cf. [Pra2]. We are thus in the tame case $(n, p) = 1$, and in this case it can be seen that Galois representations are induced from a character θ of $L^\times$ where L is a degree n extension of k, with $\theta^2 = 1$, thus θ takes values in ± 1. Therefore in this case the Galois representation is defined over $\mathbb{Q}$. Our theorem B implies that the selfdual representations of $\mathrm{GL}_m(D)$ are defined over $\mathbb{R}$, and the discussion in this section refines it to ask the following:

Question 8.2 *Is every selfdual, irreducible representation of* $\mathrm{GL}_m(D)$, *for* D/k *a division algebra of odd index, defined over* $\mathbb{Q}$ *for* m *odd?*

Recently, Bushnell and Henniart have answered this question in the affirmative in [BH] for $m = 1$ or $r = 1$.

References

[AC] J. Arthur and L. Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[Bad] A. I. Badulescu. Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations. *Invent. Math.* **172** (2008), 383–438. With an appendix by Neven Grbac.

[BH] C. Bushnell and G. Henniart. Self-dual representations of some dyadic groups, 2008.

[CC] G. Chenevier and L. Clozel. Corps de nombres peu ramifies et formes automorphes autoduales, 2007.

[Clo1] L. Clozel. On limit multiplicities of discrete series representations in spaces of automorphic forms. *Invent. Math.* **83** (1986), 265–284.

[Clo2] L. Clozel. Représentations galoisiennes associées aux représentations automorphes autoduales de $\mathrm{GL}(n)$. *Inst. Hautes Études Sci. Publ. Math.* (1991), 97–145.

[CKPSS] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi. Functoriality for the classical groups. *Publ. Math. Inst. Hautes Études Sci.* (2004), 163–233.

[CPSS] J. W. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi. Stability of γ-factors for quasi-split groups. *J. Inst. Math. Jussieu* **7** (2008), 27–66.

[GJR] D. Ginzburg, D. Jiang, and S. Rallis. On the nonvanishing of the central value of the Rankin-Selberg L-functions. *J. Amer. Math. Soc.* **17** (2004), 679–722 (electronic).

[HL] M. Harris and J.-P. Labesse. Conditional base change for unitary groups. *Asian J. Math.* **8** (2004), 653–683.

[HT] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[Hen1] G. Henniart. Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique. *Invent. Math.* **139** (2000), 439–455.

[Hen2] G. Henniart. Une caractérisation de la correspondance de Langlands locale pour $\mathrm{GL}(n)$. *Bull. Soc. Math. France* **130** (2002), 587–602.

[HH] G. Henniart and R. Herb. Automorphic induction for $\mathrm{GL}(n)$ (over local non-Archimedean fields). *Duke Math. J.* **78** (1995), 131–192.

[JS1] D. Jiang and D. Soudry. The local converse theorem for $\mathrm{SO}(2n+1)$ and applications. *Ann. of Math. (2)* **157** (2003), 743–806.

[JS2] D. Jiang and D. Soudry. Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}. In *Contributions to automorphic forms, geometry, and number theory*, pages 457–519. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[Kot] R. E. Kottwitz. Sign changes in harmonic analysis on reductive groups. *Trans. Amer. Math. Soc.* **278** (1983), 289–297.

[Lab] J.-P. Labesse. Cohomologie, stabilisation et changement de base. *Astérisque* (1999), vi+161. Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen.

[MW] C. Mœglin and J.-L. Waldspurger. Le spectre résiduel de $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)* **22** (1989), 605–674.

[Mœg] C. Mœglin. Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques. *Pacific J. Math.* **233** (2007), 159–204.

[Pra1] D. Prasad. Distinguished representations for quadratic extensions. *Compositio Math.* **119** (1999), 335–345.

[Pra2] D. Prasad. Some remarks on representations of a division algebra and of the Galois group of a local field. *J. Number Theory* **74** (1999), 73–97.

[PR] D. Prasad and D. Ramakrishnan. Lifting orthogonal representations to spin groups and local root numbers. *Proc. Indian Acad. Sci. Math. Sci.* **105** (1995), 259–267.

[Rog] J. D. Rogawski. *Automorphic representations of unitary groups in three variables*, volume 123 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1990.

[SZ] A. J. Silberger and E.-W. Zink. An explicit matching theorem for level zero discrete series of unit groups of $\mathfrak{p}$-adic simple algebras. *J. Reine Angew. Math.* **585** (2005), 173–235.

[Sou1] D. Soudry. On Langlands functoriality from classical groups to GL_n. *Astérisque* (2005), 335–390. Automorphic forms. I.

[Sou2] D. Soudry. Letter to D. Ramakrishnan, December 2007.

[Vig] M.-F. Vignéras. Correspondances entre representations automorphes de GL(2) sur une extension quadratique de GSp(4) sur **Q**, conjecture locale de Langlands pour GSp(4). In *The Selberg trace formula and related topics (Brunswick, Maine, 1984)*, volume 53 of *Contemp. Math.*, pages 463–527. Amer. Math. Soc., Providence, RI, 1986.

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