Notes on Calculus

by

Dinakar Ramakrishnan
253-37 Caltech
Pasadena, CA 91125
Fall 2001

Contents

9 Inverse Functions, log, exp, arcsin, ...

9.1 Inverse functions

Suppose f is a function with domain X and image (or range) Y. By definition, given any x in X, there is a **unique** y in Y such that $f(x) = y$. But this definition of a function is not egalitarian, because it does not require that a *unique* number x be sent to y by f; so y is special but x is not. A really nice kind of a function is what one calls a **one-to-one** (or **injective**) function. By definition, f is such a function iff

(9.1.1) $$f(x) = f(x') \implies x = x'.$$

In such a case, we can define an **inverse function** g with domain Y and range X, given by

(9.1.2) $$g(y) = x \quad \text{iff} \quad f(x) = y.$$

Clearly, when such an inverse function g exists, one has

(9.1.3) $$g \circ f = 1_X \quad \text{and} \quad f \circ g = i_Y,$$

where i_X, resp. 1_Y, denotes the **identity function** on X, resp. Y.

We will be concerned in this chapter with X, Y which are subsets of the real numbers.

Proposition 9.1.4 *Let f be a one-to-one function with domain $X \subset \mathbb{R}$ and range Y, with inverse g. Suppose in addition that f is differentiable at x with $f'(x) \neq 0$. Then g is differentiable at $y = f(x)$ and we have*

$$g'(y) = \frac{1}{f'(x)}$$

for all x in X with $y = f(x)$.

Note that if we know *a priori* that f and g are both differentiable, then this is easy to prove. Indeed, in that case their composite function $g \circ f$, which is the identity on X by (8.1.3), would be differentiable. By differentiating the identity

$$g(f(x)) = x$$

with respect to x, and applying the *chain rule*, we get

$$g'(f'(x)) \cdot f'(x) = 1,$$

because the derivative of x is 1. Done.

Proof. We have to compute the limit

$$L = \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}.$$

Since f, g are one-to-one, we can find, for each small h, a small h' such that

$$y + h = f(x + h').$$

So

$$g(y + h) - g(y) = (x + h') - x = h'$$

and

$$h = (y + h) - y = f(x + h') - f(x).$$

Moreover, h' goes to 0 as h goes to 0 (and *vice-versa*). Hence

$$L = \lim_{h' \to 0} \frac{h'}{f(x + h') - f(x)},$$

which is the inverse of $f'(x)$. It makes sense because $f'(x)$ is by assumption non-zero.

$\square$

Note that this proof shows that g is not differentiable at any point $y = f(x)$ if f' is zero at x.

It is helpful to note that many a function is not one-to-one in its maximal domain, but becomes one when restricted to a smaller domain. To give a simple example, the **squaring function**

$$f(x) = x^2$$

is defined everywhere on $\mathbb{R}$. But it is not one-to-one, because $f(a) = f(-a)$. However, if we restrict to the subset $\mathbb{R}_+$ of non-negative real numbers, f is one-to-one and so we may define its inverse to be the **square-root function**

$$g(y) = \sqrt{y}, \ \forall\, y \in \mathbb{R}_+.$$

Another example is provided by the sine function, which is periodic of period 2π and hence not one-to-one on $\mathbb{R}$. But it becomes one when restricted to $[-\pi/2, \pi/2]$.

9.2 The natural logarithm

For any $x > 0$, its natural logarithm is defined by the definite integral

$$\log x = \int_1^x \frac{dt}{t}. \tag{9.2.1}$$

Some write $\ln(x)$ instead, and some others write $\log_e x$. When $0 < x < 1$, this signifies the negative of integral of $\frac{1}{t}$ from x to 1. Consequently, $\log x$ is positive if $x > 0$, negative if $x < 0$ and equals 0 at $x = 1$.

Proposition 9.2.2

(a) $\log 1 \;=\; 0$.

(b) $\log x$ *is differentiable everywhere in its domain* $\mathbb{R}_+ = (0, \infty)$ *with derivative* $\frac{1}{x}$.

(c) (**addition theorem**) *For all* $x, y > 0$,

$$\log(xy) \;=\; \log x + \log y.$$

(d) $\log x$ *is a strictly increasing function.*

(e) $\log x$ *becomes unbounded in the positive direction when* x *goes to* ∞ *and it is unbounded in the negative direction when* x *goes to* 0.

(f) $\log x$ *is integrable on any finite subinterval of* $\mathbb{R}_+$, *and its indefinite integral is given by*

$$\int \log x \, dx \;=\; x \log x - x + C.$$

(g) $\log x$ *goes to* $-\infty$ *(resp.* ∞*) slowly when* x *goes to* 0 *(resp.* ∞*); more precisely,*

(i)
$$\lim_{x \to 0} x \log x \;=\; 0$$

and

(ii)
$$\lim_{x \to \infty} \frac{\log x}{x} \;=\; 0.$$

(h) The **improper integral** *of* $\log x$ *over* $(0, b]$ *exists for any* $b > 0$, *with*

$$\int_0^b \log x \, dx \;=\; b \log b - b.$$

Property (c) is very important, because it can be used to **transform multiplicative problems into additive ones**. This was the motivation for their introduction by **Napier** in 1616. Property (b) is also important. Indeed, if we assume only the properties (a),(c),(d),(e) for a function f on $\mathbb{R}_+$, there are lots of functions (*logarithms*) which satisfy these properties. But the situation becomes rigidified with a **unique solution** once one requires (b) as well. This is why log is called the **natural logarithm**. The other (unnatural) choices will be introduced towards the end of the next section.

Proof. (**a**): Since $\log 1 = 0$ and $\exp \circ \log$ is the identity function,

$$\exp(0) \;=\; \exp(\log(1)) \;=\; 1.$$

(**b**): For any x, the function $\frac{1}{t}$ is continuous on $[0, x]$, hence by the *First Fundamental Theorem of Calculus*, $\log x$ is differentiable at x with derivative $\frac{1}{x}$.
(**c**): Fix any $y > 0$ and consider the function of x defined by

$$\ell(x) \;=\; \log(xy) \quad \text{on} \quad \{x > 0\}.$$

Since it is the composite of the differentiable functions $x \to xy$ and $u \to \log u$, ℓ is also differentiable. Applying the *chain rule*, we get

$$\ell'(x) \;=\; y\left(\frac{1}{yx}\right) \;=\; \frac{1}{x}.$$

Thus ℓ and log both have the same derivative, and so their difference must be independent of x. Write

$$\ell(x) \;=\; \log x + c.$$

Evaluating at $x = 1$, and noting that $\log 1 = 0$ and $\lambda(1) = \log x$, we see that c must be $\log x$, proving the *addition formula*.
(**d**): As we saw in chapter 6, a differentiable function f is strictly increasing iff its derivative is positive everywhere. When $f(x) = \log x$, the derivative, as we saw above, is $1/x$, which is positive for $x > 0$. This proves (d).
(**e**): As $\log x$ is strictly increasing and since it vanishes at 1, its value at any number $x_0 > 1$, for instance at $x_0 = 2$, is positive. By the addition theorem and induction, we see that for any positive integer n,

$$\log(x_0^n) \;=\; n \log x_0. \tag{9.2.3}$$

Consequently, as n goes to ∞, $\log(x_0^n)$ goes to ∞ as well. This proves that $\log x$ is unbounded in the positive direction. For the negative direction, note that for any positive $x_1 < 1$, $\log x_1$ is negative (since $\log 1 = 0$ and $\log x$ is increasing). Applying (8,2,3) with x_0 replaced by x_1, we deduce that $\log(x_1^n)$ goes to $-\infty$ as n goes to ∞. Done.
(**f**) Since $\log x$ is continuous, it is integrable on any finite interval in $(0, \infty)$. Moreover, by *integration by parts*,

$$\int \log x \, dx \;=\; x \log x - \int x \frac{d}{dx}(\log x) dx.$$

The assertion (f) now follows since the expression on the right is $x \log x - x + C$.
(**g**): Put

$$L \;=\; \lim_{x \to 0} x \log x$$

and

$$u \;=\; \frac{1}{x},\; f(u) \;=\; -\log\left(\frac{1}{u}\right), \quad \text{and} \quad g(u) \;=\; u.$$

Then we have

$$L \;=\; -\lim_{u \to \infty} \frac{f(u)}{g(u)}.$$

Then by (e), $f(u)$ and $g(u)$ approach ∞ as u goes to ∞, and both these functions are differentiable at any positive u. (All one needs is that they are differentiable for large u.) Since $u'(x) = -\frac{1}{x^2} = -u^2$ and $f(u) = -\log x$, we have by the *chain rule*,

$$f'(u) \;=\; \frac{df/dx}{du/dx} \;=\; \frac{-1/x}{-u^2} \;=\; \frac{1}{u}.$$

Since $g'(u) = 1$ for all u, we then get

$$\lim_{u\to\infty} \frac{f'(u)}{g'(u)} \;=\; 0.$$

So we may apply L'Hopital's rule (see the **Appendix** to this chapter), and conclude that

$$L \;=\; -\lim_{u\to\infty} \frac{f'(u)}{g'(u)} \;=\; 0,$$

giving (i). The proof of (ii) is similar and will be left for the reader to check.
(**h**): The improper integral of $\log x$ exists over $(0, b]$ iff the following limit exists:

$$L \;=\; \lim_{x\to 0} \int_x^b \log t\, dt.$$

Thanks to (f), we have

$$\int_x^b \log t\, dt \;=\; (t\log t - t)|_x^b \;=\; (b\log b - b) - x\log x + x.$$

To prove (h) we need to show that

$$\lim_{x\to 0} (x\log x - x) \;=\; 0,$$

which is a consequence of (g).

□

9.3 The exponential function

In view of the discussion in section 9.1, and the fact (parts (d), (e) of Proposition 9.2.2) that $\log x$ is a strictly increasing function with domain $\mathbb{R}_+$ and range $\mathbb{R}$, we can define the **exponential function**, exp(**x**) for short, to be the inverse function of $\log x$. Note that $\exp(x)$ has domain $\mathbb{R}$ and range $\mathbb{R}_+$.

Proposition 9.3.1

(a) $\exp(0) = 1$.

(b) $\exp(x)$ *is differentiable everywhere in* $\mathbb{R}$ *with derivative* $\exp(x)$.

(c) *(***addition theorem***)* *For all* $x, y \in \mathbb{R}$,

$$\exp(x + y) = \exp(x)\exp(y).$$

(d) $\exp(x)$ *is a strictly increasing function.*

(e) $\exp(x)$ *becomes unbounded in the positive direction when* x *goes to* ∞ *and it goes to* 0 *when* x *goes to* $-\infty$.

(f) $\exp(x)$ *is integrable on any finite subinterval of* $\mathbb{R}_+$, *and its indefinite integral is given by*

$$\int \exp(x)\,dx = \exp(x) + C.$$

(g) *As* x *goes to* ∞ *(resp.* $-\infty$*),* $\exp(x)$ *goes to* ∞ *(resp.* 0*) faster than any polynomial* $p(x)$ *goes to* ∞ *(resp.* $-\infty$*), i.e.,*

(i)
$$\lim_{x\to\infty} \frac{p(x)}{\exp(x)} = 0$$

and

(ii)
$$\lim_{x\to-\infty} p(x)\exp(x) = 0.$$

(h) *The* **improper integral** *of* $\exp(x)$ *over* $(-\infty, b]$ *exists for any* $b > 0$, *with*

$$\int_{-\infty}^{b} \exp(x)\,dx = \exp(b).$$

Proof. (**a**): This is immediate from the definition.
(**b**): Note that the derivative $\frac{1}{x}$ of $\log x$ is nowhere zero on its domain $\mathbb{R}_+$. So we may apply Proposition 9.1.4 with $f = \log$, $g = \exp$ and $y = \log x$ to get the everywhere differentiability of exp, with

$$\frac{d}{dy}\exp(y) = \frac{1}{(\log x)'} = \frac{1}{1/x} = x = \exp(y).$$

(**c**): Fix x, y in $\mathbb{R}$. Then we can find (unique) u, v in $\mathbb{R}_+$ such that

$$x = \log u \quad \text{and} \quad y = \log v.$$

Applying part (c) of Proposition 9.2.2, we then obtain

$$\exp(x+y) \;=\; \exp(\log u + \log v) \;=\; \exp(\log(uv)) \;=\; uv.$$

The assertion follows because $u = \exp(x)$ and $v = \exp(y)$.
(**d**): Let $x > y$ be arbitrary in $\mathbb{R}$. Write $x = \log u$, $y = \log v$ as above. Since log is strictly increasing, we need $u > v$. Since $u = \exp(x)$ and $v = \exp(v)$, we are done.
(**e**): Suppose $\exp(x)$ is bounded from above as x goes to infinity, i.e., suppose there is a positive number M such that $\exp(x) < M$ for all $x > 0$. Then, since log is a strictly increasing function, $x = \log(\exp(x))$ would be less than $\log(M)$ for all positive x, which is absurd. So $\exp(x)$ must be unbounded as x becomes large. To show that $\exp(x)$ approaches 0 as x goes to $-\infty$, we need to show that for any $\varepsilon > 0$, there exists $-T < 0$ such that

$$\exp(x) \;<\; \varepsilon \quad \text{whenever} \quad x < -T. \tag{9.3.2}$$

Applying the logarithm to both inequalities, writing $-T = \log u$ for a unique $u \in (0,1)$ and $\varepsilon' = \exp(\varepsilon)$, and using the fact that log is a one-to-one function, we see that (9.3.2) is equivalent to the statement

$$x \;<\; \varepsilon' \quad \text{whenever} \quad \log x < \log u, \tag{9.3.3}$$

which evidently holds by the properties of the logarithm. Hence $\exp(x)$ goes to 0 as x goes to $-\infty$.
(**f**) Since $\exp(x)$ is continuous, it is integrable on any finite subinterval of $\mathbb{R}$. In fact, since $\exp(x)$ is its own derivative, it is its own **primitive** as well, proving the assertion.
(**g**): To prove (i), it suffices to show that for any $j \geq 0$, the limit

$$L \;=\; \lim_{x\to\infty} x^j \exp(x)$$

exists and equals 0. When $j = 0$ this follows from (e), while the assertion for $j > 1$ follows from the case $j = 1$, which we will assume to be the case from here on. It then suffices to show that the function $\exp(x)/x$ goes to ∞ as x goes to ∞. By taking logarithms, this becomes equivalent to the statement that $x/\log x$ goes to ∞ as $\log x$, and hence x, goes to ∞, which is what we proved in our proof of the limit (i) of Proposition 9.2.2, part (g). (One can also apply, with care, L'Hopital's rule directly to the quotient $x/\exp(x)$ and thereby establish (i).) The proof of (ii) is similar and will be left for the reader to check.
(**h**): The improper integral of $\exp(x)$ exists over $(-\infty, b]$ iff the following limit exists:

$$L \;=\; \lim_{x\to-\infty} \int_x^b \exp(t)dt.$$

Thanks to (f), we have

$$\int_x^b \exp(t)\,dt \;=\; \exp(b) - \exp(x).$$

To prove (h) we need only show that

$$\lim_{x \to -\infty} \exp(x) = 0,$$

which is a consequence of (e).

□

How many differentiable functions are there which are derivatives (and hence primitives) of themselves? This is answered by the following

Lemma 9.3.4 *Let f be any differentiable function on an open interval (a, b), possibly of infinite length, satisfying $f'(x) = f(x)$ for all x in (a, b). Then there exists a scalar c such that*

$$f(x) = c \exp(x) \; \forall \, x \in (a, b).$$

Moreover, if $f(0) = 1$, then $f(x)$ equals $\exp(x)$ on this interval.

Proof. Define a function h on (a, b) by

$$h(x) = \frac{f(x)}{\exp(x)},$$

which makes sense because exp never vanishes anywhere.

Since f and exp are differentiable, so is h, and by the *quotient rule* we have

$$h'(x) = \frac{f'(x)\exp(x) - f(x)\exp'(x)}{\exp(x)^2} = \frac{f(x)\exp(x) - f(x)\exp(x)}{\exp(x)^2} = 0,$$

because f and exp are derivatives of themselves. Therefore $h(x)$ must be a constant c, say. The Lemma now follows.

□

Definition 9.3.5 *Put*

$$e = \exp(1).$$

Equivalently, e is the unique, positive real number whose natural logarithm is 1. It is not hard to see that, to a first approximation, $2 < e < 4$. One can do much better with some work, of course, and also show that e is **irrational**, even **transcendental**.

A natural question now arises. We have worked with the power function a^x before, for any positive real number a, which satisfies the same addition rule as $\exp x$, i.e.,

$$a^{x+y} = a^x \cdot a^y. \tag{9.3.6}$$

. What is the relationship between e^x and $\exp(x)$. The answer is very satisfying.

Proposition 9.3.7 *For all x in $\mathbb{R}$, we have*

$$\exp(x) = e^x.$$

We will for now accept the following result, which will be proved later in chapter 11:

Lemma 9.3.8 *We have*

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Proof of Proposition 9.3.7. First we claim that e^x is its own derivative. Indeed, by (9.3.6),

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right).$$

The right hand side limit is e^x by Lemma 9.3.8, yielding the claim.

Next we apply Lemma 9.3.4 to deduce that e^x must be a multiple of $\exp(x)$. In fact, since $e^0 = 1$, e^x must equal $\exp(x)$.

□

Just like $\exp(x)$, the power function a^x is, for any positive number a, a strictly increasing function. One calls the inverse of a^x by the name **logarithm to the base** a and denotes it by $\log_a y$. Then $\log_a$ satisfies many of the properties of log, but its derivative is not $1/x$ (if $a \neq e$.

Proposition 9.3.9 *Fix* $a > 0$. *Then*

$$\frac{d}{dx}(a^x) = a^x \log a,$$

and

$$\frac{d}{dy}(\log_a y) = \frac{1}{y \log a}.$$

Proof. Note that

$$a^x = \exp(\log(a^x)) = e^{x \log a},$$

which is the composite of $x \to x \log a$ and $u \to e^u$. Applying the chain rule, we obtain

$$\frac{d}{dx}(a^x) = e^{x \log a} \cdot \log a = a^x \log a.$$

The formula for the derivative of $\log_a y$ follows from this and Proposition 9.1.4.

□

The basic **hyperbolic functions** are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh x}{\cosh x}. \tag{9.3.10}$$

The other hyperbolic functions are given by the inverses of these.

9.4 arcsin, arccos, arctan, et al

The sine function, which is defined and differentiable everywhere on $\mathbb{R}$, is periodic with period 2π and is therefore not a one-to-one function. It however becomes one when the domain is restricted to $[-\pi/2, \pi/2]$. One calls the corresponding inverse function the **arcsine** function, denoted $\arcsin x$. Clearly, the domain of $\arcsin x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. Moreover, since the derivative $\cos x$ of $\sin x$ is non-zero, in fact positive, in $(-\pi/2, \pi/2)$, we may apply Proposition 9.1.4 and deduce that the arcsine function is differentiable on $(-\pi/2, \pi/2)$, with

$$\frac{d}{dy}(\arcsin y) \;=\; \frac{1}{\cos x} \;=\; \frac{1}{\sqrt{1-\sin^2 x}} \;=\; \frac{1}{1-y^2}, \tag{9.4.1}$$

where $y = \sin x$.

One notes similarly that $\cos x$, resp. $\tan x$, is one-to-one on $[0, \pi]$, resp. $(-\pi/2, \pi/2)$, and defines its inverse function $\arccos x$, resp. $\arctan x$, with domain $[-1, 1]$ and range $[0, \pi]$. Arguing as above, we see that $\arccos x$ is differentiable on $(0, \pi)$ and $\arctan x$ is differentiable on $(-\pi/2, \pi/2)$, with

$$\frac{d}{dy}(\arccos x) \;=\; -\frac{1}{\sqrt{1-y^2}} \tag{9.4.2}$$

and

$$\frac{d}{dy}(\arctan x) \;=\; \frac{1}{1+y^2} \tag{9.4.3}$$

Consequently, we obtain

$$\int \frac{dy}{1+y^2} \;=\; \arctan y + C \tag{9.4.4}$$

and

$$\int \frac{dy}{\sqrt{1-y^2}} \;=\; \arcsin y + C. \tag{9.4.5}$$

This incidentally brings to a close our quest to integrate arbitrary rational functions, which we began in the previous chapter, where we reduced the problem to the evaluation of the integral on the left of (9.4.4).

It should be notes that it is a miracle that we can evaluate the reciprocal of the square root of $1 - y^2$ (for $-1 \leq y \leq 1$) in terms of $\arcsin y$. If one tried to integrate $\frac{1}{\sqrt{f(y)}}$ for a polynomial f of degree $n > 2$, the problem becomes forbiddingly difficult. Even for $n = 3$, one needs to use **elliptic functions**.

9.5 A useful substitution

A very useful substitution to deal with trigonometric integrals is to set

$$u = \tan(x/2), \tag{9.5.1}$$

which implies that

$$x = 2\arctan u.$$

Note that

$$\frac{du}{dx} = \frac{1}{2}\sec^2(x/2)\,\frac{1}{2}(1+\tan^2(x/2)) = \frac{1+u^2}{2}, \tag{9.5.2}$$

$$dx = \frac{2}{1+u^2}du, \tag{9.5.3}$$

$$\sin x = 2\sin(x/2)\cos(x/2) = 2\frac{\tan(x/2)}{\sec^2(x/2)} = \frac{2u}{1+u^2}, \tag{9.5.4}$$

and since $\cos^2(x/2)+\sin^2(x/2)=1$,

$$\cos x = \frac{\cos^2(x/2)-\sin^2(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-u^2}{1+u^2}. \tag{9.5.5}$$

For example, suppose we have to integrate

$$I = \int \frac{dx}{1-\sin x}.$$

Using the substitution above, which is justifiable here, we get

$$I = \int \frac{1}{1-2u/(1+u^2)}\frac{2}{1+u^2}du = \int 2\frac{du}{1-2u+u^2}.$$

Since

$$\frac{1}{1-2u+u^2} = \frac{1}{(1-u)^2} = \frac{d}{du}\left(\frac{1}{1-u}\right),$$

we get

$$I = \int d\left(\frac{1}{1-u}\right) = \frac{1}{1-u} + C = \frac{1}{1-\tan(x/2)} + C.$$

Appendix: L'Hopital's Rule

The following result is often helpful but should be used with care, making sure that the hypotheses are satisfied in any particular situation of interest.

Theorem (**L'Hopital's rule**) *Consider a limit of the form*

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)},$$

where f, g are differentiable functions in some (a, ∞), such that both and f and g approach ∞ as x goes to ∞. Then L exists **if** *the* **limit quotient of the derivatives**, *namely*

$$L' = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

exists. Moreover, when L' exists, L equals L'.

A proof of this will hopefully be posted at a later date. In the meanwhile, the students are urged to understand the statement carefully and make use of it where applicable.