Notes on Calculus

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8 Factorization of polynomials, Integration by partial fractions

We know by now how to integrate any polynomial. What about rational functions? This is much harder. To approach it systematically, we first need some basic facts about polynomials and their factorizations. This leads to what is called a **partial fraction expansion** of the reciprocal $\frac{1}{f(x)}$ of a polynomial $f(x)$. Using this we can reduce the study of the general case to the following three cases:

(8.0.1) $$\int \frac{dx}{(x-a)^m}$$

(8.0.2) $$\int \frac{dx}{(x^2+bx+c)^m}$$

and

(8.0.3) $$\int \frac{xdx}{(x^2+bx+c)^m}.$$

Here a, b, c are arbitrary scalars and m any positive integer.

It turns out that we can compute (8.0.1) and (8.0.3) without trouble (see section 8.4) by using *substitution* and the logarithm. To evaluate (8.0.2) even for $m = 1$, however, will require that we understand the *arctangent* function, which we will be do in chapter 9. One tackles the $m > 1$ case of (8.1.2) by a reduction process, as in the evaluation of the integral of $\cos^n x$.

8.1 Long division, roots

Suppose F is any field, for example $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$. By a **polynomial of degree n with coefficients in F**, we mean a function of the form

(8.1.1) $$f(x) \;=\; a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \quad \text{with} \quad a_j \in F\;(\forall j) \quad \text{and} \quad a_n \neq 0.$$

One writes $\deg(f)$ for the degree n of f. Note that if two g is a polynomial of degree m with coefficients in F, then

(8.1.2) $$\deg(fg) \;=\; \deg(f) + \deg(g),$$

while

$$\deg(f+g) \;\leq\; \max\{\deg(f), \deg(g)\}.$$

One defines the **degree of the zero polynomial** to be $-\infty$. This is forced upon us so as to have (8.1.2) when $f = 0$ and g arbitrary. (Think about it!)

Everyone is used to doing **long division** of polynomials in High School, and what results is the following useful result, called the **Euclidean algorithm for polynomials**:

Theorem 8.1.3 *Let f, g be polynomials with coefficients in F. Then we can find other polynomials Q, called the* **quotient polynomial**, *and R, called the* **remainder**, *such that*

$$f(x) = Q(x)g(x) + R(x) \quad \text{with} \quad \deg(R) < \deg(g).$$

We will say that g **divides** f, or that g **is a factor of** f, when there is no remainder, i.e., when R is the zero polynomial, in which case we have the **factorization**

$$f(x) = Q(x)g(x).$$

A number α in F will be called a **root** of a polynomial f iff we have $f(\alpha) = 0$. The following useful Lemma tells us the relationship between having a root and being factorizable.

Lemma 8.1.4 *Given a scalar α in F and a non-constant polynomial f with coefficients in F, the folowing are equivalent:*

(i) α is a root of f;

(ii) $x - \alpha$ divides $f(x)$.

Proof. Applying Theorem 8.1.3 with $g(x) = x - \alpha$, we get

$$f(x) = Q(x)(x - \alpha) + R(x) \tag{8.1.5}$$

with $\deg(R) < \deg(g) = 1$. Then the only possibility is for R to be a **constant polynomial**, i.e., a scalar c. Then, plugging in $x = \alpha$ in (8.1.5), we obtain

$$f(\alpha) = Q(\alpha)(\alpha - \alpha) + c = c,$$

which implies that $f(\alpha)$ is zero iff c is zero, which happens iff $x - \alpha$ divides f.

□

To paraphrase this Lemma, **every time we find a root we can factor!**

A polynomial f will be called **irreducible over** F iff f has **no factor** g with

$$0 < \deg(g) < \deg(f). \tag{8.1.5}$$

Lemma 8.1.6

(i) Any **linear polynomial** *f, i.e., one of degree 1, is irreducible.*

(ii) The polynomial $f(x) = ax^2 + bx + c$ is irreducible over $\mathbb{R}$ iff the **discriminant** *$D = b^2 - 4ac$ is negative.*

The most important example of (ii) is of course $x^2 + 1$.

Proof. Part (i) is clear from the condition (8.1.5). So let us prove part (ii). We know from High School mathematics that the roots of the quadratic polynomial $f(x) = ax^2+bx+c$ are given by

$$\alpha_{\pm} \;=\; -\frac{-b \pm \sqrt{D}}{2a},$$

with $D = b^2 - 4ac$. These roots always exist in $\mathbb{C}$. But for them to exist in $\mathbb{R}$, it is necessary and sufficient that D have a square-root in $\mathbb{R}$, i.e., that D be non-negative in $\mathbb{R}$. (The roots are not distinct iff $D = 0$.)

8.2 Factorization over $\mathbb{C}$

The most important result over $\mathbb{C}$, which is the reason people are so interested in working with complex numbers, is the following:

Theorem 8.2.1 (**The Fundamental Theorem of Algebra**) *Every non-constant polynomial with coefficients in $\mathbb{C}$ admits a root in $\mathbb{C}$.*

We will not prove this result here. But one should become aware of its existence if it is not already the case! We will now give an important consequence.

Corollary 8.2.2 Let f be a polynomial of degree $n \geq 1$ with $\mathbb{C}$-coefficients. Then there exist complex numbers $\alpha_1, \ldots, \alpha_r$, with $\alpha_i \neq \alpha_j$ if $i \neq j$, positive integers $m_1, \ldots, m_r$, and a scalar c, such that

$$f(x) \;=\; c\prod_{j=1}^{r}(x - \alpha_j),$$

and

$$\sum_{j=1}^{r} m_j \;=\; n.$$

In other words, any non-constant polynomial f with $\mathbb{C}$-coefficients **factorizes completely into a product of linear factors**. For each $j \leq r$, the associated positive integer m_j is called the **multiplicity** of α_j as a root of f, which means concretely that m_j is the highest power of $(x - \alpha_j)$ dividing $f(x)$.

Proof of Corollary. Let $n \geq 1$ be the degree of f and let a_n be the non-zero **leading coefficient**, i.e, the coefficient of x^n (see (8.1.1)). Let us set

$$c \;=\; a_n. \tag{8.2.3}$$

If $n = 1$,

$$f(x) \;=\; a_1x + a_0 \;=\; c(x - \alpha_1) \quad \text{with} \quad \alpha_1 \;=\; \frac{a_0}{a_1}.$$

So we are done in this case by taking $r = 1$ and $m_1 = 1$.

Now let $n > 1$ and assume by induction that we have proved the assertion for all $m < n$, in particular for $m = n - 1$. By Theorem 8.2.1, we can find a root, call it β, of f. By applying (8.1.4), we may then write

$$f(x) \;=\; (x - \beta)h(x), \tag{8.2.4}$$

for some polynomial $h(x)$ necessarily of degree $n - 1$. The leading coefficients of f are evidently the same. By induction we may write

$$h(x) \;=\; c\prod_{i=1}^{s}(x - \alpha_i)^{k_i},$$

for some roots $\alpha_1, \ldots, \alpha_s$ of h with respective multiplicities $n_1, \ldots, n_s$, so that

$$\sum_{i=1}^{s} k_i \;=\; n - 1.$$

But by (8.2.4), every root of h is also a root of f, and the assertion of the Corollary follows. □

8.3 Factorization over $\mathbb{R}$

The best way to understand polynomials f with real coefficients is to first look at their complex roots and then determine which ones of them could be real. To this end recall first the baby fact that a complex number $z = u + iv$ is real iff z equals its **complex conjugate** $\overline{z} = u - iv$, where $i = \sqrt{-1}$.

Proposition 8.3.1 *Let*

$$f(x) \;=\; a_0 + a_1x + \ldots + a_nx^n \quad \text{with} \quad a_j \in \mathbb{R}\,\forall j \leq n \quad \text{and} \quad a_n \neq 0,$$

for some $n \geq 1$. Suppose α is a **complex root** *of f. Then $\overline{\alpha}$ is also a root of f. In particular, if r denotes the number of real roots of f and s the non-real (complex) roots of f, then we must have*

$$n \;=\; r + 2s.$$

We get the following consequence, which we proved earlier using the *Intermediate value theorem.*

Corollary 8.3.2 *Let f be a real polynomial of odd degree. Then f must have a real root.*

Proof of Proposition. Let α be a complex root of f. Recall that for all complex numbers z, w,

$$\overline{zw} \;=\; \overline{z}\overline{w} \quad \text{and} \quad \overline{z + w} \;=\; \overline{z} + \overline{w}. \tag{8.3.3}$$

Hence for any $j \leq n$,

$$(\overline{\alpha})^j = \overline{\alpha^j}.$$

Moreover, since $a_j \in \mathbb{R}$ $(\forall j)$, $\overline{a}_j = a_j$, and therefore

$$a_j(\overline{\alpha})^j = \overline{a_j\alpha^j}.$$

Consequently, using (8.3.3) again, we get

$$f(\overline{\alpha}) = \sum_{j=0}^{n} a_j(\overline{\alpha})^j = \overline{f(\alpha)}. \tag{8.3.4}$$

But α is a root of f (which we have not used so far), $f(\alpha)$ vanishes, as does its complex conjugate $\overline{f(\alpha)}$. So by (8.3.4), $f(\overline{\alpha})$ is zero, showing that $\overline{\alpha}$ is a root of f.

So the **non-real roots** come in **conjugate pairs**, and this shows that n minus the number r, say, of the **real roots** is even. Done.

□

Given any complex number z, we have

$$z + \overline{z},\ z\overline{z} \in \mathbb{R}. \tag{8.3.5}$$

This is clear because both the **norm** $z\overline{z}$ and the **trace** $z + \overline{z}$ are unchanged under complex conjugation.

Proposition 8.3.6 *Let f be a real polynomial of degree $n \geq 1$ with real roots $\alpha_1, \ldots, \alpha_k$ with multiplicities $n_1, \ldots, n_k$, and non-real roots $\beta_1, \overline{\beta}_1, \ldots, \beta_\ell, \overline{\beta}_\ell$ with multiplicities $m_1, \ldots, m_\ell$ in $\mathbb{C}$. Then we have the factorization*

$$f(x) = c\prod_{i=1}^{k}(x-\alpha_i)^{n_i} \cdot \prod_{j=1}^{\ell}(x^2 + b_jx + c_j)^{m_j}, \tag{$*$}$$

where for each $j \leq \ell$,

$$b_j = -(\beta_j + \overline{\beta}_j) \quad \text{and} \quad c_j = \beta_j\overline{\beta}_j,$$

Each of the factors occurring in $(*)$ is a real polynomial, and the polynomials $x - \alpha_i$ and $x^2 + b_jx + c_j$ are all irreducible over $\mathbb{R}$.

Proof. In view of Corollary 8.2.2 and Proposition 8.3.1, the only thing we need to prove is that for each $j \leq \ell$, the polynomial

$$h_j(x) = x^2b_j + c_j$$

is real and irreducible over $\mathbb{R}$. The reality of the coefficients $b_j = -(\beta_j + \overline{\beta}_j)$ and $c_j = \beta_j\overline{\beta}_j$ follows from (8.3.5). Suppose it is reducible over $\mathbb{R}$. Then we can write

$$h_j(x) = (x - t_j)(x - t_j')$$

for some real numbers t_j, t_j'. On the other hand $\beta_j, \overline{\beta}_j$ are roots of h_j. This forces the equality of the sets $\{t_j, t_j'\}$ and $\{\beta_j, \overline{\beta}_j\}$, contradicting the fact that β_j is non-real. So h_j must be irreducible over $\mathbb{R}$.

□

8.4 The partial fraction decomposition

Here is the main result.

Theorem 8.4.1 *Let*

$$g(x) \;=\; \prod_{i=1}^{k}(x-\alpha_i)^{n_i}\cdot\prod_{j=1}^{\ell}(x^2+b_jx+c_j)^{m_j},$$

where the α_i, b_j, c_j are real, and the n_i, m_j are positive integers. Then there exist real numbers $A_i^{(p)}, B_j^{(q)}, C_j^{(q)}$, with $1 \le i \le k$, $1 \le p \le n_i$, $1 \le j \le \ell$ and $1 \le q \le m_j$, such that

$$\frac{1}{g(x)} \;=\; \sum_{i=1}^{k}\sum_{p=1}^{n_i}\frac{A_j^{(p)}}{(x-\alpha_i)^{(p)}} \;+\; \sum_{j=1}^{\ell}\sum_{q=1}^{m_j}\frac{B_j^{(q)}x+C_j^{(q)}}{(x^2+b_jx+c_j)^{(q)}}. \tag{8.4.2}$$

We will not prove this here. But here is the basic idea of the proof. Cross multiply (8.4.2) and get a polynomial equation of degree $n = \sum_{i=1}^{k} n_i + \sum_{j=1}^{\ell} m_j$ (which is the degree of g) where the coefficients involve the n indeterminates $A_i^{(p)}, B_j^{(q)}, C_j^{(q)}$. One solves for them by comparing the coefficients of x^i, for $1 \le i \le n$. This results in an $n \times n$ **linear system**, i.e., a system of n linear equations in the n unknowns. In Ma1b you will learn to determine when such a linear system has a solution.

Let us try to understand this procedure in the simple case when

$$g(x) \;=\; (x-\alpha)^2(x^2+bx+c).$$

We want to show that there exist numbers A^1, A^2, B, C such that

$$\frac{1}{g(x)} \;=\; \frac{A^{(1)}}{x-\alpha} + \frac{A^{(2)}}{(x-\alpha)^2} + \frac{Bx+C}{x^2+bx+c}.$$

Cross multiplying, this gives the equation

$$1 \;=\; A^{(1)}(x-\alpha)(x^2+bx+c) + A^{(2)}(x^2+bx+c) + (Bx+C)(x-\alpha)^2.$$

Multiplying the right hand side out, we obtain

$$1 \;=\; A^{(1)}(x^3+(b-\alpha)x^2+(c-\alpha)x-c\alpha)+A^{(2)}(x^2+bx+c)+B(x^3-2\alpha x^2+\alpha^2x)+C(x^2-2\alpha x+\alpha^2).$$

Comparing coefficients, we get

$$(i) \qquad A^{(1)}+B \;=\; 0,\; A^{(1)}(b-\alpha)+A^{(2)}-2B\alpha+C \;=\; 0,$$

$$(ii) \qquad A^{(1)}(c-\alpha)+A^{(2)}b+B\alpha^2-2C\alpha \;=\; 0, \quad \text{and } -A^{(1)}c\alpha+A^{(2)}c+C\alpha^2 \;=\; 1.$$

This gives four linear equations in four unknowns, namely in $A^{(1)}, A^{(2)}, B$ and C. The equations (i) imply

$$(iii) \qquad A^{(1)} = -B \quad \text{and} \quad A^{(2)} = -A^{(1)}(b-\alpha) + 2B\alpha - C = B(b+\alpha) - C.$$

Eliminating $A^{(1)}, A^{(2)}$ from (ii) using (iii), we get

$$(iv) \qquad B(\alpha^2 + (b+1)\alpha + b^2 - c) - C(b+2\alpha) = 0$$

and

$$(v) \qquad B(b+2\alpha)c + C(\alpha^2 - c) = 1.$$

It can be checked that the linear equations (iv), (v) are independent, so that we can solve for B, C in terms of α, b, c. Then we can find $A^{(i)}, i = 1, 2$ by using (iii).

To have a **numerical example**, take

$$\alpha = 1, b = 0, c = 1.$$

Then (iv) becomes $B - 2C = 0$ and (v) becomes $2B = 1$, so the solution we seek is given by

$$B = \frac{1}{2},\ C = \frac{1}{4},\ A^{(1)} = -\frac{1}{2},\ A^{(2)} = \frac{1}{2}.$$

Therefore

$$\frac{1}{(x-1)^2(x^2+1)} = -\frac{1}{2(x-1)} + \frac{1}{2(x-1)^2} + \frac{2x+1}{4(x^2+1)}.$$

8.5 Integration of rational functions

Let us begin discussing a simple situation. The numerical example at the end of section 8.4 implies, by the additivity of the integral, that

$$(8.5.1) \qquad \int \frac{dx}{(x-1)^2(x^2+1)} = I_1 + I_2 + I_3,$$

where

$$I_1 = -\frac{1}{2}\int \frac{dx}{x-1},$$

$$I_2 = \frac{1}{2}\int \frac{dx}{(x-1)^2}$$

and

$$I_3 = \frac{1}{4}\int \frac{2x+1}{x^2+1}dx.$$

We will see in the next chapter that

$$I_1 = -\frac{1}{2}\log|x-1| + C.$$

Using substitution and the knowledge of the integral of x^t, we get

$$I_2 \;=\; -\frac{1}{2(x-1)} + C.$$

And

$$I_3 \;=\; I_{3,1} + I_{3,2},$$

where

$$I_{3,1} \;=\; \frac{1}{4}\int \frac{2x}{x^2+1}dx \;=\; \frac{1}{4}\log(x^2+1) + C,$$

which was evaluated by using the substitution $u = x^2 + 1$, and

$$I_{3,2} \;=\; \frac{1}{4}\int \frac{dx}{x^2+1},$$

which we will be able to evaluate in the next chapter when we study the arctan function.

Suppose we want to integrate a **general rational function**. We have the following result.

Proposition 8.5.2 *Let $\frac{f(x)}{g(x)}$ be a rational function, i.e., a quotient of polynomials $f(x), g(x)$, with real coefficients. Then the (indefinite)*

$$I = \int \frac{f(x)}{g(x)}dx$$

can be written as a real linear combination of integrals of the form (8,0,1), (8,0,2) and (8,0,3) and the integral of a polynomial (which shows up only when deg(f) $\geq$deg(g)).

Proof. Thanks to Proposition 8.3.6 and Theorem 8.4.1, we can write I as a linear combination of integrals of the form

$$I_1 \;=\; \int \frac{h(x)}{(x-a)^m}dx \tag{8.5.3}$$

and

$$I_2 \;=\; \int \frac{h(x)}{(x^2+bx+c)^m}dx,$$

where $h(x)$ denotes a polynomial with real coefficients. In fact, $h(x)$ is in I_1, resp. I_2, a multiple of $f(x)$, resp. $(Ax+B)f(x)$ for some A, B.

There is nothing to prove if $f(x)$ is a constant. So let us take the degree of f to be ≥ 1. The Proposition is clearly a consequence of the following

Lemma 8.5.4 *Let $\phi(x)$ be a real polynomial of degree ≥ 1, and let a, b, c be real numbers with $a, b \neq 0$. Then*

(i) We can write $\phi(x)$ as a polynomial in $(x-a)$ with real coefficients.

(ii) If $\phi(x)$ has degree ≥ 2, then we can write

$$\phi(x) \;=\; \sum_{j=0}^{r} \lambda_j(x)(bx+c)^j,$$

where each $\lambda_j(x)$ is a real polynomial of degree ≤ 1.

Proof of Lemma 8.5.4. Let the degree of $\phi(x)$ be n.
(i) The assertion is obvious if $n = 1$, So take n to be > 1 and assume by induction that the assertion holds for $n-1$. Using Theorem 8.1.3, we can write

$$\phi(x) \;=\; Q(x)(x-a) + c_1, \tag{8.5.5}$$

where c_1 is a constant. Since $Q(x)$ is of degree $n-1$, we may apply the inductive hypothesis and conclude that $Q(x)$ is a real polynomial in $x-a$. Then (8.5.5) shows that $\phi(x)$ is also a polynomial in $x-a$, as claimed. Done.
(ii) We will apply the *Principle of Induction* to the set of all integers ≥ 2. Suppose $n = 2$, with $\phi(x) = Ax^2+Bx+C$, $A \neq 0$. Then $\phi(x)$ can be written as $A(x^2+bx+c)+((B-Ab)x+(C-Ac))$, so the assertion holds with $\lambda_0(x) = (B-Ab)x+(C-Ac)$ and $\lambda_1(x) = A$. So take n to be greater than 2 and assume by induction that the assertion holds for all $m < n$. Now applying Theorem 8.1.3 again, we may write

$$\phi(x) \;=\; Q(x)(x^2+bx+c) + \lambda_0(x), \tag{8.5.6}$$

where $\lambda_0(x)$ is a real polynomial of degree < 2. Since the degree of $Q(x)$ is of degree $n-2$, we may apply the inductive hypothesis and conclude that

$$Q(x) \;=\; \sum_{i=0}^{k} \mu_i(x)(x^2+bx+c),$$

with each $\mu_i(x)$ if a real polynomial of degree ≤ 1. Combining this with (8.5.6), we get what we want with $r = k+1$ and $\lambda_j(x) = \mu_{j-1}(x)$ for each $j \geq 1$.

□

The next key step is to **complete the square**. Explicitly, we can, given any pair of real numbers b, c, write

$$x^2+bx+c \;=\; (x+\frac{b}{2})^2 + (c-\frac{b^2}{4}). \tag{8.5.7}$$

When x^2+bx+c is irreducible over $\mathbb{R}$, which we may assume to be the case, thanks to Proposition 8.3.6, its discriminant b^2-4c is necessarily negative, and so $c-\frac{b^2}{4}$ is positive and thus can be expressed as e^2, for a positive real number e. Consequently, by using the substitution $u = x+\frac{b}{2}$, which gives $u'(x) = 1$, we can transform the integrals (8.0.2) and (8.0.3) into linear combinations of integrals of the following form:

$$I_1 \;=\; \int \frac{du}{(u^2+e^2)^m} \tag{8.5.8}$$

and

$$I_2 = \int \frac{udu}{(u^2+e^2)^m}.$$

There is no problem at all in evaluating I_2. If we put $v = u^2 + e^2$, then $dv = 2udu$, and

$$I_2 = \frac{1}{2}\int \frac{dv}{v^m},$$

which equals

$$\frac{1}{2}\log|v| + C = \frac{1}{2}\log|u^2+e^2| + C \quad \text{if} \quad m = 1,$$

and

$$-\frac{1}{2(m-1)v^{m-1}} + C = -\frac{1}{2(m-1)(u^2+e^2)^{m-1}} + C \quad \text{if} \quad m > 1.$$

We may use substitution again to simplify I_1. Indeed, if we set $y = u/e$, we have

$$dy = \frac{1}{e}du, \quad \text{and} \quad u^2 + e^2 = e^2(y^2+1).$$

Consequently,

$$I_1 = \frac{1}{e}\int \frac{dy}{(y^2+1)^m}. \tag{8.5.9}$$

As mentioned above, this will turn out to be given by $\frac{1}{e}\arctan y + C$ when $m = 1$.

It is left to discuss a **reduction process** which allows us to compute

$$J_m = \int \frac{dy}{(y^2+1)^m}$$

for $m > 1$.

Let us try the substitution

$$y = \tan t.$$

Since $\tan^2 t + 1 = \sec^2 t$, we have

$$(y^2+1)^m = \sec^{2m} t \quad \text{and} \quad dy = \sec^2 t dt.$$

Consequently, since $\frac{1}{\sec t} = \cos t$,

$$J_m = \int \cos^{2(m-1)} t\, dt.$$

We have already evaluated it in the previous chapter. Finally, to write the answer in the variable y, we will need to write t as $\arctan y$.