Notes on Calculus

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Contents

7 The Fundamental Theorems of Calculus, Methods of Integration

So far we have separately learnt the basics of integration and differentiation. But they are not unrelated. In fact, they are **inverse operations**. This is what we will try to explore in the first section, via the two fundamental theorems of Calculus. After that we will discuss the two main methods one uses for integrating somewhat complicated functions, namely **integration by substitution** and **integration by parts**.

7.1 The fundamental theorems

Suppose f is an integrable function on a closed interval $[a, b]$. Then we can consider the **signed area function A** on $[a, b]$ (relative to f) defined by the definite integral of f from a to x, i.e.,

$$A(x) = \int_a^x f(t)dt. \tag{7.1.0}$$

The reason for the **signed area** terminology is that f is not assumed to be ≥ 0, so apriori $A(x)$ could be negative.

It is extremely interesting to know how $A(x)$ varies with x. What conditions does one need to put on f to make sure that A is continuous, or even differentiable? The continuity part of the question is easy to answer.

Lemma 7.1.1 *Let f, A be as above. Then A is a continuous function on $[a, b]$.*

Proof. Let c be any point in $[a, b]$. Then f is continuous at c iff we have

$$\lim_{h\to 0} A(c+h) = A(c).$$

Of course, i taking the limit, we consider all small enough h for which $c+h$ lies in $[a, b]$, and then let h go to zero. By the *additivity* of the integral, we have (using (7.1.10)),

$$A(c+h) - A(c) = \int_{I(c,h)} f(t)dt,$$

where $I(c, h)$ denotes the closed interval between c and $c + h$. Clearly, $I(c, h)$ is $[c, c + h]$, resp. $[c + h, c]$, if h is positive, resp. negative. When h goes to zero, $I(c, h)$ shrinks to the point $\{c\}$, and so

$$\lim_{h\to 0} A(c+h) - A(c) = 0,$$

which is what we needed to show.

$\square$

The question of differentiability of A is more subtle, and the complete answer is given by the following important result:

Theorem 7.1.2 (**The first fundamental theorem of Calculus**) *Let f be an integrable function on $[a,b]$, and let A be the function defined by (7.1.0). Pick any point c in (a,b), and suppose that f is continuous at c. Then A is differentiable there and moreover,*

$$A'(c) \;=\; f(c).$$

Some would write this symbolically as

$$\frac{d}{dx}\int_a^x f(t)dt \;=\; f(x). \tag{7.1.3}$$

In plain words, this says that *differentiating the integral gives back the original* as long as the original function is continuous at the point in question.

Proof. To know if $A(x)$ is differentiable at c, we need to evaluate the limit

$$L \;=\; \lim_{h\to 0}\frac{A(c+h)-A(c)}{h}. \tag{7.1.4}$$

By the *additivity* of the definite integral, we have

$$A(c+h)-A(c) \;=\; \int_{I(c,h)} f(x)dx, \tag{7.1.5}$$

where $I(c,h)$ is as in the proof of Lemma 7.1.1.

Denote by $M(c,h)$, resp. $m(c,h)$, the *supremum*, resp. *infemum*, of the values of f over $I(c,h)$. Then the following bounds evidently hold:

$$hm(c,h) \;\le\; \int_{I(c,h)} f(x)dx \;\le\; hM(c,h). \tag{7.1.6}$$

Combining (7.1.4), (7.1.5) and (7.1.6), we get for all small h,

$$\lim_{h\to 0} m(c,h) \;\le\; L \;\le\; \lim_{h\to 0} M(c,h). \tag{7.1.7}$$

But by hypothesis, f is continuous at c. Then both $m(c,h)$ and $M(c,h)$ will tend to $f(c)$ as h goes to 0, which proves the Theorem in view of (7.1.7).

□

Let f be any function on an open interval I. Suppose there is a differentiable function ϕ on I such that $\phi'(x) = f(x)$ for all x in I. Then we will call ϕ a **primitive** of f on I. Note that the primitive is not unique. Indeed, for any constant α, the function $\phi + \alpha$ will have the same derivative as ϕ. Intuitively, one feels immediately that the notion of a primitive should be tied up with the notion of an integral. The following very important and oft-used result makes this expected relationship precise.

Theorem 7.1.8 (**The second fundamental theorem of Calculus**) *Suppose f, ϕ are functions on $[a, b]$, with f integrable on $[a, b]$ and ϕ a primitive of f on (a, b), with ϕ defined and continuous at the endpoints a, b. Then*

$$\phi(b) - \phi(a) = \int_a^b f(x)dx.$$

One can rewrite this, perhaps more expressively, as

$$\phi(b) - \phi(a) = \int_a^b \frac{d}{dx}\phi dx.$$

Proof. Choose any partition

$$P: \ a = t_0 < t_1 < \ldots < t_n = b,$$

and set, for each $j \in \{1, 2, \ldots, n\}$,

$$M_j = \sup(f([t_{j-1}, t_j])) \quad \text{and} \quad m_j = \inf(f([t_{j-1}, t_j])).$$

By definition,

$$\sum_{j=1}^{n}(t_j - t_{j-1})m_j \le \int_a^b f(x)dx \le \sum_{j=1}^{n}(t_j - t_{j-1})m_j. \tag{7.1.9}$$

On the other hand, the **Mean Value Theorem** gives us, for each j, a number c_j in $[t_{j-1}, t_j]$ such that

$$\phi'(c_j) = \frac{\phi(t_j) - \phi(t_{j-1})}{t_j - t_{j-1}}. \tag{7.1.10}$$

Since ϕ is by hypothesis the *primitive* of f on (a, b), $f(c_j) = \phi'(c_j)$ for each j. Moreover,

$$m_j \le f(c_j) \le M_j, \tag{7.1.11}$$

and

$$\sum_{j=1}^{n} \phi(t_j) - \phi(t_{j-1}) = \phi(b) - \phi(a). \tag{7.1.12}$$

Combining (7.1.10), (7.1.11) and (7.1.12), we obtain

$$\sum_{j=1}^{n}(t_j - t_{j-1})m_j \le \phi(b) - \phi(a) \le \sum_{j=1}^{n}(t_j - t_{j-1})m_j. \tag{7.1.13}$$

Since (7.1.9) and (7.1.13) hold for *every* partition P, and since f is integrable on $[a, b]$, the assertion of the Theorem follows.

$\square$

7.2 The indefinite integral

Suppose ϕ is a primitive of a function f on an open interval I. It is not unusual to set, following Leibniz,

$$\int f(x)dx = \phi(x). \tag{7.2.1}$$

This is called an **indefinite integral** because there are no limits and ϕ is non-unique. So one can think of such an indefinite integral as a function of x which is unique only up to addition of an arbitrary constant. One has, in other words, an equality for all scalars C

$$\int f(x)dx = \int f(x)dx + C.$$

It could be a bit unsettling to work with such an indefinite, nebulous function at first, but one learns soon enough that it is a useful concept to be aware of.

In many Calculus texts one finds formulas like

$$\int \cos x dx = \sin x + C$$

and

$$\int \frac{1}{x}dx = \log x + C.$$

All they mean is that $\sin x$ and $\log x$ are the primitives of $\cos x$ and $\frac{1}{x}$, i.e.,

$$\frac{d}{dx}\sin x = \cos x$$

and

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

Of course the situation is completely different in the case of definite integrals.

7.3 Integration by substitution

There are a host of techniques which are useful in evaluating various definite integrals. We will single out two of them in this chapter and analyze them. The first one is the method of substitution, which one should always try first before trying others.

Theorem 7.3.1 *Let $[a, b]$ be a closed interval and g a function differentiable on an open interval containing $[a, b]$, with g' continuous on $[a, b]$. Also let f be a continuous function on $g([a, b])$. Then we have the identity*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof. Let ϕ denote a primitive of f, which exists because the continuity assumption on f makes it integrable on $[a, b]$. Then we have, by the second fundamental theorem of Calculus,

$$\int_{g(a)}^{g(b)} f(u)du \;=\; \phi(g(b)) - \phi(g(a)) \;=\; (\phi \circ g)(b) - (\phi \circ g)(a). \tag{7.3.2}$$

On the other hand, by the *chain rule* applied to the composite function $\phi \circ g$, we have

$$(\phi \circ g)'(x) \;=\; (\phi' \circ g)(x) \cdot g'(x) \;=\; (f \circ g)(x) \cdot g'(x).$$

Consequently,

$$\int_a^b f(g(x))g'(x)dx \;=\; \int_a^b (\phi \circ g)'(x)dx. \tag{7.3.3}$$

Applying the second fundamental theorem of Calculus again, the right hand side of (7.3.3) is the same as

$$(\phi \circ g)(b) - (\phi \circ g)(a). \tag{7.3.4}$$

The Theorem now follows by combining (7.3.2), (7.3.3) and (7.3.4).

$\square$

Before giving some examples let us note that powers of $\sin x$ and $\cos x$, as well as polynomials, are differentiable on $\mathbb{R}$ with continuous derivatives. In fact we can differentiate them any number of times; one says they are *infinitely differentiable.* The same holds for ratios of such functions or their combinations, as long as the denominator is non-zero in the interval of interest.

Examples 7.3.5: **(1)** Let

$$I \;=\; \int_0^{\pi/2} sin^3 x \cos x dx.$$

Thanks to the remark above on the *infinite differentiability* of the functions in the *integrand*, we are allowed to apply Theorem 7.3.1 here, with

$$g(x) \;=\; \sin x \quad \text{and} \quad f(u) = u^3.$$

Then, since $g'(x) = \cos x$ (as proved earlier, $g(0) = 0$ and $g(\pi/2) = 1$, we obtain

$$I \;=\; \int_0^1 u^3 du \;=\; \frac{1}{4}.$$

(2) Put

$$I = \int_0^{\pi/4} \cos^2 x dx.$$

Recall that

$$\cos^2 x - \sin^2 x = \cos 2x.$$

Since $\cos^2 x + \sin^2 x = 1$, we get

$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$

Using this and the easy integral $\int_0^{\pi/4} dx = \pi/4$, we get

$$I = \frac{\pi}{8} + J, \quad \text{with} \quad J = \frac{1}{2}\int \cos 2x dx.$$

Put $g(x) = 2x$ and $f(u) = \cos u$, which are both infinitely differentiable on all of $\mathbb{R}$, and use Theorem 7.3.1. to conclude that, since $g'(x) = 2$, $g(0) = 0$ and $g(\pi/4) = \pi/2$,

$$J = \frac{1}{4}\int_0^{\pi/4} \cos u du = \frac{\sin(\pi/4) - \sin 0)}{4} = \frac{1}{4\sqrt{2}},$$

This implies that

$$I = \frac{\pi}{8} + \frac{1}{4\sqrt{2}} = \frac{\pi + \sqrt{2}}{8}.$$

(3) Evaluate

$$I = \int_0^1 \sqrt{1 - x^2} dx.$$

Here we use the substitution theorem in the reverse direction. The basic idea is that $\sqrt{1 - x^2}$ would simplify if x were $\sin t$ or $\cos t$. Put

$$g(t) = \sin t \quad \text{and} \quad f(u) = \sqrt{u}.$$

Then g is differentiable everywhere with $g'(t) = \cos t$ being continuous on $[0, 1]$. We chose the interval $[0, 1]$ because $g(0) = 0$ and $g(\pi/2) = 1$, giving us the limits of integration of I. Also, f is continuous on $g([0, \pi/2]) = [0, 1]$. (At the end point 0, the continuity of f means it is right continuous there. This is good, because f is not defined to the left of 0.) So we have satisfied all the hypotheses of Theorem 7.3.1 and we may apply it to get

$$I = \int_0^{\pi/2} f(g(t))g'(t)dt = \int_0^{\pi/2} \sqrt{1 - \sin^2 t}\cos t dt.$$

But
$$\sqrt{1-\sin^2 t} \;=\; \sqrt{\cos^2 t} \;=\; |\cos t|,$$
which is just $\cos t$, because the cosine function is non-negative in the interval $[0, \pi/2]$. Hence
$$I \;=\; \int_0^{\pi/2} \cos^2 t dt.$$
We just evaluated the integral of $\cos^2 t$ in the previous example, albeit with different limits. In any case, proceeding as in that example, we get
$$I \;=\; \frac{\pi}{4} + \frac{\sin(\pi/2) - \sin 0}{4} \;=\; \frac{\pi - 1}{4}.$$

7.4 Integration by parts

Some consider this the most important theorem of Calculus. Its use is pervasive.

Theorem 7.4.1 *Let $[a,b]$ be a closed interval and let f, g be differentiable functions in an open interval around $[a,b]$ such that f', g' continuous on $[a,b]$. Then we have*
$$\int_a^b f(x)g'(x)dx \;=\; f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx.$$

Here $f(x)g(x)|_a^b$ denotes $f(b)g(b) - f(a)g(a)$.

Proof. By the *product rule*,

$$(fg)'(x) \;=\; f(x)g'(x) + f'(x)g'(x) \tag{7.4.2}$$

for all x where f and g are both differentiable. Subtracting $f'(x)g(x)$ from both sides and integrating over $[a,b]$ we get

$$\int_a^b f(x)g'(x)dx \;=\; \int_a^b (fg)'(x)dx - \int_a^b f'(x)g(x)dx. \tag{7.4.3}$$

But by the second fundamental theorem of Calculus,

$$\int_a^b (fg)'(x)dx \;=\; (fg)(b) - (fg)(a). \tag{7.4.4}$$

The assertion now follows by combining (7.4.3) and (7.4.4).

$\square$

Example 7.4.5: Evaluate, for *any* integer $n \geq 0$,

$$I_n = \int_0^{\pi/2} \cos^n x dx.$$

First note that

$$I_0 = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2}$$

and

$$I_1 = \int_0^{\pi/2} \cos x dx = \sin(\pi/2) - \sin 0 = 1,$$

because $\sin(\pi/2) = 1$ and $\sin 0 = 0$.

We have already solved the $n = 2$ case in the previous section using substitution, but we will not use it here. Integration by parts is more powerful!

So we may suppose that $n > 1$. Put

$$f(x) = \cos^{n-1} x \quad \text{and} \quad g(x) = \sin x.$$

Then, as noted earlier, f and g are infinitely differentiable on all of $\mathbb{R}$, with

$$f'(x) = (n-1)\cos^{n-2} x \cdot (-\sin x) \quad \text{and} \quad g'(x) = \cos x,$$

where the first formula comes from the chain rule. Now we may apply Theorem 7.4.1 and obtain

$$I_n = \left(\cos^{n-1}(x) \sin x\right)|_0^{\pi/2} - (n-1) \int_0^{\pi/2} cos^{n-2} x(-\sin x) \cdot \sin x dx.$$

Note that $n - 1 \neq 0$ as $n > 1$ and $\cos\theta \sin\theta$ is 0 if θ is 0 or $\pi/2$. Therefore the first term on the right is zero, and we get

$$I_n = (n-1) \int_0^{\pi/2} \cos^{n-2} x \sin^2 x dx.$$

Since $\sin^2 x = 1 - \cos^2 x$, we get

$$I_n = (n-1)I_{n-2} + (n-1)I_n,$$

which translates into the neat **recursive relation**

$$I_n = \frac{n-1}{n} I_{n-2}.$$

In particular,

$$I_2 = \frac{1}{2}I_0 = \frac{\pi}{4},\ I_4 = \frac{3}{4}I_2 = \frac{3\pi}{16}, \ldots$$

and

$$I_3 = \frac{2}{3}I_1 = \frac{2}{3},\ I_5 = \frac{4}{5}I_3 = \frac{8}{15}.$$

It will be left as a nice exercise for the reader to find exact formulae for I_{2n} and I_{2n-1}.