Notes on Calculus

by

Dinakar Ramakrishnan
253-37 Caltech
Pasadena, CA 91125
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5 Improper Integrals, Areas, Polar Coordinates, Volumes

We have now attained a good level of understanding of integration of nice functions f over closed intervals $[a, b]$. In practice one often wants to extend the domain of integration and consider unbounded intervals such as $[a, \infty)$ and $(-\infty, b]$. The simplest non-trivial examples are the **infinite trumpets** defined by the areas under the graphs of x^t for $t > 0$, i.e., the *improper integrals*

$$A_t = \int_1^\infty \frac{1}{x^t} dx.$$

We will see below that A_t has a well defined meaning if $t > 1$, but becomes unbounded for $t \leq 1$.

One is also interested in integrals of functions f over finite intervals with f being unbounded. The natural examples are given (for $t > 0$) by

$$B_t = \int_0^1 \frac{1}{x^t} dx.$$

Here it turns out that B_t is well defined, i.e., has a finite value, if and only if $t < 1$. In particular, neither A_1 nor B_1 makes sense. Moreover,

$$A_t + B_t = \int_0^\infty \frac{1}{x^t} dx$$

is unbounded for every $t > 0$.

After we learn the fundamental theorems of Calculus, we will visit the land of improper integrals again with new tools.

5.1 Improper Integrals

Let f be a function defined on the interior of a possibly infinite interval J such that either its upper endpoint – call it b, is ∞ or f becomes unbounded as one approaches b. But suppose that the lower endpoint – call it a, is finite and that $f(a)$ is defined. In the former case the interval is unbounded, while in the latter case the interval is bounded, but the function is unbounded. We will say that the **integral of f over J** exists iff the following two conditions hold:
(5.1.1)

(i) For every $u \in (a, b)$, f is integrable on $[a, u]$; and

(ii) the limit

$$\lim_{u\to b,\, u<b} \int_a^u f(x)dx$$

exists.

When this limit exists, we will call it the **integral of f over J** and write it symbolically as

$$\int_a^b f(x)dx.$$

Similarly, if a is either $-\infty$ or is a finite point where f becomes unbounded, but with b a finite point where f is defined, one sets

$$\int_a^b f(x)dx = \lim_{u\to a,\, u>a} \int_u^b f(x)dx \tag{5.1.2}$$

when the limit on the right makes sense.

Lemma 5.1.3 *We have*

$$\int_a^\infty f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx \quad \forall\, c \in (a,\infty)$$

and

$$\int_{-\infty}^b f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^b f(x)dx \quad \forall\, c \in (-\infty, b),$$

when the integrals make sense.

Proof. We will prove the first **additivity formula** and leave the other as an exercise for the reader. Pick any $c \in (a,\infty)$. or all real numbers $u > c$, we have by the usual additivity formula,

$$\int_a^u f(x)dx = \int_a^c f(x)dx + \int_c^u f(x)dx.$$

Thus

$$\int_a^\infty f(x)dx = \lim_{u\to\infty} \left(\int_a^c f(x)dx + \int_c^u f(x)dx \right),$$

which equals

$$\int_a^c f(x)dx + \lim_{u\to\infty} \int_c^u f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx.$$

$\square$

If J is an interval with both of its endpoints being problematic, we will choose a point c in (a, b) and put

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{5.1.4}$$

if both the improper integrals on the right make sense. One can check using Lemma 5.1.3 above that this definition is independent of the choice of c.

When an improper integral does not make sense, we will call it **divergent**. Otherwise it is **convergent**.

Proposition 5.1.4 *Let t be a positive real number. Then for $t > 1$,*

$$\int_1^\infty \frac{1}{x^t}dx = \frac{1}{t-1}, \tag{A}$$

with the improper integral on the left being divergent for $t \leq 1$.

On the other hand, if $t \in (0, 1)$,

$$\int_0^1 \frac{1}{x^t}dx = \frac{1}{1-t}, \tag{B}$$

with the improper integral on the left being divergent for $t \geq 1$.

Proof. We learnt in chapter 3 that for all $a, b \in \mathbb{R}$ with $a < b$, and for all $t \neq 1$:

$$\int_a^b \frac{1}{x^t}dx = \frac{b^{1-t} - a^{1-t}}{1-t}.$$

Hence for $t > 1$,

$$\lim_{u\to\infty} \int_1^u \frac{1}{x^t}dx = \lim_{u\to\infty} \frac{1}{(1-t)u^{t-1}} + \frac{1}{t-1} = \frac{1}{t-1},$$

because the term $\frac{1}{u^{t-1}}$ goes to zero as u goes to ∞. If $t < 1$, this term goes to ∞ as u goes to ∞, and so the integral is divergent. Finally let $t = 1$. Then we cannot use the above formula. But for each $N \geq 1$, we have the inequality

$$\sum_{n=1}^N \frac{1}{n} \leq \int_1^N \frac{1}{x}dx.$$

The reason is that the sum on the left is a lower Riemann sum for the function $f(x) = \frac{1}{x}$ over the interval $[1, N]$ relative to the partition $P : 1 < 2 < \ldots < N$. So, if the improper integral of this function exists over $[1, \infty)$, the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n}$$

must converge. But we have seen in chapter 2 that this series diverges. So the integral is divergent for $t = 1$, and (A) is proved in all cases.

The proof of (B) is very similar and will be left as an exercise.

□

5.2 Areas

Given any region R in the plane, one is often interested in assigning, if possible, an **area**. Whenever the area makes sense, R will be called a **measurable region**. If the region is bounded, one can proceed in the following intuitive way. For each $n > 1$, enclose R in a rectangular grid made up of squares of side $1/n$. Add up the areas of all the squares in the grid which meet R, and denote this (upper) sum by $U_n(R)$. Then add up the areas of all the squares which are contained in R, and call this (lower) sum $L_n(R)$. Then let n become very large and see if the two sums converge to the same limit. In the event they do, we will say that R is **measurable** and call the common limit the **area of** R, denoted by $A(R)$. Of course this procedure should be reminiscent of the way we evaluated the integral of a nice function f on a closed interval $[a, b]$. Indeed, the definite integral was defined in such a way that when it exists and when f is non-negative on $[a, b]$, its value will equal the area under the graph of f over $[a, b]$. To be precise we should say that the value is the area of the region caught between the graph of f over $[a, b]$, the x-axis and the two vertical lines $x = a$ and $x = b$.

Now that we have defined improper integrals, we can also consider **areas of infinite plane regions** and define them by a **limiting process**. But we should note that to get a finite answer, the region will need to get narrower and narrower as the x-coordinate gets near ∞ or $-\infty$. Similarly, we can consider regions of bounded width, but unbounded vertically; for the area to exist here, the region will need to get very narrow as the y-coordinate gets near ∞ or $-\infty$.

Here are some basic properties of the area function of measurable sets in the plane: (5.2.1)

(i) $A(R) \geq 0$.

(ii) $A(R \cup R') = A(R) + A(R') - A(R \cap R')$.

(ii) $A(R)$ does not change if R gets translated or rotated in the plane.

Proposition 5.2.2 *Let J be an interval, possibly of infinite length. Let f, g be functions which are defined on the interior of J and are integrable on J. Denote by R the region between the graph of f, graph of g, and the vertical lines at the end points of J unless J is of infinite length, in which case one takes the vertical lines only at the finite endpoint if any. Then R is measurable and*

$$A(R) = \int_J |f-g|(x)dx,$$

where $|f-g|$ denote the function $x \mapsto |f(x)-g(x)|$.

For example let us calculate $A(R)$ when $J=[0,\pi/2]$, $f(x)=\cos x$ and $g=\sin x$. Then $f-g$ is non-negative in $[0,\pi/4]$ and non-positive in $[\pi/4,\pi/2]$. So we get, by the additivity of the integral,

$$A(R) = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx.$$

Recall that for all a, b with $a<b$,

$$\int_a^b \sin x dx = \cos a - \cos b$$

and

$$\int_a^b \cos x dx = \sin b - \sin a.$$

So by the linearity of the integral,

$$A(R) = \{(\sin\pi/4 - \sin 0) - (\cos 0 - \cos\pi/4)\} + \{(\cos\pi/4 - \cos\pi/2) - (\sin\pi/2 - \sin\pi/4)\}.$$

One knows that $\sin 0 = \cos\pi/2 = 0$, $\cos 0 = \sin\pi/2 = 1$, and $\sin\pi/4 = \cos\pi/4 = 1/\sqrt{2} = \sqrt{2}/2$. Hence

$$A(R) = \frac{\sqrt{2}}{2} - 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 + \frac{\sqrt{2}}{2} = 2(\sqrt{2}-1).$$

Sometimes one speaks of the area of the region R between the graphs of two functions without specifying the base interval. In that case R is taken to be the bounded region, if any, caught between the graphs. If the graphs do not meet or if they meet just once, then there is no bounded region.

For *example*, consider the functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = x^3 - x.$$

To find the bounded region R caught between the graphs, one has to first solve for the points (if any) where the graphs meet. This is the same as finding the points x where $x^2 = x^3 - x$, i.e., $x = 0$ or $x^2 - x - 1 = 0$. There are three real solutions, namely $x = 0, \frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Note that $x^3 - x \geq x^2$ on $[\frac{1-\sqrt{5}}{2}, 0]$, and $x^3 - x \leq x^2$ on $[0, \frac{1+\sqrt{5}}{2}]$. (Sketch the graphs and see!) So we get

$$A(R) = \int_{(1-\sqrt{5})/2}^{0} (x^3 - x - x^2)dx - \int_0^{(1+\sqrt{5})/2} (x^3 - x - x^2)dx.$$

Since $\int_a^b x^n dx$ is, for $n \neq -1$, $(b^{n+1} - a^{n+1})/(n+1)$,

$$A(R) = -\frac{(1-\sqrt{5})^4}{64} + \frac{(1-\sqrt{5})^2}{8} + \frac{(1-\sqrt{5})^3}{24} + \frac{(1+\sqrt{5})^4}{64} - \frac{(1+\sqrt{5})^2}{8} - \frac{(1+\sqrt{5})^3}{24}.$$

This simplifies to give

$$A(R) = \frac{5\sqrt{5}}{12}.$$

5.3 Polar coordinates

So far we have confined ourselves to **rectilinear coordinates** on the plane, which are often called **Cartesian coordinates** to honor René Descartes who introduced them. Simply put, we identify each point P on the plane by the pair (x, y), where x (resp. y) is the distance between the origin O and the point where the x-axis (resp. y-axis) meets the perpendicular to it from P. Instead one can look at the pair (r, θ), where r is the distance from P to the origin, measured on the line L connecting O to P, and the angle between the x-axis and L, measured in the counterclockwise direction. By definition $r \geq 0$, and we will take θ to lie in $[0, 2\pi)$. In particular, the angle is taken to be zero, and not 2π or 4π or -2π, for any point lying on the x-axis. It should also be noted that as defined, the angle does not make much sense for the origin; we take (r, θ) to be $(0, 0)$ for it.

The quantities r, θ are called the **polar coordinates** of P. It is easy to see that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{5.3.1}$$

In the reverse direction, one can almost recover (r, θ) from (x, y) by the easily verified formulae

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}. \tag{5.3.2}$$

We said *almost*, because we do not know at the moment to what extent $\tan\theta$ determines θ. We will come back to this question after studying inverse functions.

Sometimes the equation defining a curve, or a region, in the plane becomes simpler when we use polar coordinates, and this is the reason for studying them. For example, the circle centered at O defined by the equation $x^2 + y^2 = a^2$ can be easily described as the graph of $r = a$. The region inside the circle is simply $r \leq a$. The rule of thumb is that whenever there is **circular symmetry** in a given situation, it is better to use polar coordinates.

Suppose S is an **angular sector**, i.e., the region bounded by $\theta = a, \theta = b$ and $r = \rho$, with $b - a \in [0, 2\pi]$. ρ is called the **radius**, and $b - a$ the **angle**, of s. Using the definition of π as the area of the region inside the unit circle, one can show

$$A(S) \;=\; \frac{1}{2}(b-a)\rho^2. \tag{5.3.3}$$

We will accept this as a basic fact.

The main problem here will be to understand the **radial sets**, and to know when they are measurable. Such a set is given as the region bounded by $\theta = a, \theta = b$ and $r = f(\theta)$, where f is a function of $[a, b]$ and $b - a \in [0, 2\pi]$. One can use Calculus to find the area of R under a hypothesis on f.

Let us call a function f on $[a, b]$ **square-integrable** iff f^2 is integrable on $[a, b]$. Note that every continuous function on $[a, b]$ is square-integrable.

Proposition 5.3.4 *Let R be a radial set bounded by $\theta = a, \theta = b$ and $r = f(\theta)$, where f is a square-integrable function of $[a, b]$ and $b - a \in [0, 2\pi]$. Then*

$$A(R) \;=\; \frac{1}{2}\int_a^b f(\theta)d\theta.$$

Proof. Pick any partition

$$P:\; a = t_0 < t_1 < \ldots < t_n = b.$$

For every integer j with $1 \leq j \leq n$, write

$$m_j \;=\; \inf f([t_{j-1}, t_j]) \quad \text{and} \quad M_j \;=\; \sup f([t_{j-1}, t_j]), \tag{5.3.5}$$

and denote by S_j, resp. S'_j, the angular sector of radius m_j, resp. M_j, between $\theta = t_{j-1}$ and $\theta = t_j$. Then

$$\frac{1}{2}L(f^2, P) \;=\; \frac{1}{2}\sum_{j=1}^{n}(T_j - t_{j-1})m_j^2 \;=\; \sum_{j=1}^{n} A(S_j) \tag{5.3.6}$$

and

$$\frac{1}{2}U(f^2, P) \;=\; \frac{1}{2}\sum_{j=1}^{n}(T_j - t_{j-1})M_j^2 \;=\; \sum_{j=1}^{n} A(S'_j).$$

It follows that

$$\frac{1}{2}L(f^2, P) \;\leq\; A(R) \;\leq \frac{1}{2}U(f^2, P). \tag{5.3.7}$$

Since f^2 is integrable on $[a, b]$ by hypothesis, the upper sums and lower sums (of f^2) converge to a common limit, which is the integral of f^2 over $[a, b]$. Now the assertion of the Proposition follows by virtue of (5.3.7).

$\square$

As an **example**, let us look at the region R bounded by the **spiral of Archimedes**:

(5.3.8) $$0 \leq \theta \leq 2\pi,\ r = \theta.$$

Since $f(\theta) = \theta$ is square-integrable we may apply Proposition 5.3.4 and deduce that

$$A(R) \;=\; \frac{1}{2}\int_0^{2\pi} \theta^2 d\theta \;=\; \frac{1}{2}\frac{(2\pi)^3}{3} \;=\; \frac{4\pi^3}{3}.$$

5.4 Volumes

Given a three-dimensional shape V, a basic question which arises is whether we could assign a **volume** to it. When V is bounded one could give an intuitive definition, much like we did for the area of a bounded surface. For every $n > 1$, enclose V in a *three-dimensional grid* consisting of cubes whose sides are of length $1/n$. Define the upper sum U_n, resp. the lower sum L_n, to be the sum of the volumes of all the cubes which meet V, resp. all the cubes which are enclosed in V. As n increases, one gets a further refinement of the grid. We will say that V has a volume, or that V is a **measurable subset of three-space** iff the sequences $\{U_n\}$ and $\{L_n\}$ both converge and have a common limit, which will be denoted as $\mathrm{vol}(V)$.

It is not hard to see that V is itself a **cube**, then its volume is a^3 if a is the length of any **edge** of V. It can also be shown that if V has **constant thickness**, then its volume is the area of the cross-section times the thickness. In particular, the volume of any **cylinder** V of thickness h and **circular cross-section** of radius ρ, $\mathrm{vol}(V)$ is $\pi\rho^2 h$. Going one step further, if V is a **cylindrical shell** of thickness h and **annular cross-section** of **outer radius** ρ_1 and **inner radius** ρ_2, then $\mathrm{vol}(V)$ is $\pi(\rho_1^2 - \rho_2^2)h$. We will need to use these facts below.

The three-dimensional shapes which we can try to understand right now using just one-variable Calculus are the ones obtained by revolving about either axis a plane region under the graph of a (one-variable) function.

Proposition 5.4.1 *Let f be a non-negative, square-integrable function on a closed interval $[a, b]$, and let R be the region under the graph of f between $x = a$ and $x = b$. Denote by V the three-dimensional space obtained by revolving R about the x-axis. Then V is a measurable subset of three-space and moreover,*

$$\mathrm{vol}(V) \;=\; \pi\int_a^b f(x)^2 dx.$$

Proof. Choose any partition

$$P: \; a = t_0 < t_1 < \ldots < t_n = b,$$

and for every integer j between 1 and n, let m_j, M_j be as in (5.3.5). We get two cylinders of thickness $t_j - t_{j-1}$ and circular cross-sections of radii m_j, M_j. Clearly we have

$$\pi \sum_{j=1}^{n} (t_j - t_{j-1}) m_j^2 \;\leq\; \mathrm{vol}(V) \;\leq\; \pi \sum_{j=1}^{n} (t_j - t_{j-1}) M_j^2.$$

The expression on the left (resp. right) is just the lower (resp. upper) sum $L(\pi f^2, P)$ (resp. $U(\pi f^2, P)$).

Since f^2 is by assumption integrable on $[a, b]$, the least upper bound $\underline{I}(\pi f^2)$ of $\{L(\pi f^2, P)\}$ equals the greatest lower bound $\overline{I}(\pi f^2)$ of $\{U(\pi f^2, P)\}$. On the other hand, the bounds above imply that

$$\underline{I}(\pi f^2) \;\leq\; \mathrm{vol}(V) \;\leq \overline{I}(\pi f^2).$$

The assertion now follows.

□

When we revolve such an R about the y-axis, it becomes even more interesting.

Proposition 5.4.2 *Let f be a function on a closed interval $[a, b]$ with $a \geq 0$, and let R be the region caught between the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. Denote by W the three-dimensional space obtained by revolving R about the y-axis. If the function $x \mapsto xf(x)$ is integrable on $[a, b]$, then W is a measurable subset of three-space and moreover,*

$$\mathrm{vol}(W) \;=\; 2\pi \int_a^b xf(x)dx.$$

Proof. For any partition

$$P: \; a = t_0 < t_1 < \ldots < t_n = b,$$

and for every integer j between 1 and n, we get two cylindrical shells of thickness m_j, M_j and annular cross-section of inner radius t_{j-1} and outer radius t_j. Then we get

$$\pi \sum_{j=1}^{n} (t_j^2 - t_{j-1}^2) m_j \;\leq\; \mathrm{vol}(W) \;\leq\; \pi \sum_{j=1}^{n} (t_j^2 - t_{j-1}^2) M_j.$$

Since $t_j^2 - t_{j-1}^2 = (t_j - t_{j-1})(t_j + t_{j-1})$, we get

(5.4.3) $$\pi(A_1(P) + A_2(P)) \;\leq\; \mathrm{vol}(W) \;\leq\; \pi(A_3(P) + A_4(P)),$$

where

$$A_1(P) \;=\; \sum_{j=1}^{n} (t_j - t_{j-1}) t_j m_j,$$

$$A_2(P) = \sum_{j=1}^{n} (t_j - t_{j-1})t_{j-1}m_j,$$

$$A_3(P) = \sum_{j=1}^{n} (t_j - t_{j-1})t_j M_j,$$

and

$$A_4(P) = \sum_{j=1}^{n} (t_j - t_{j-1})t_{j-1}M_j.$$

We now **claim** that for every $\varepsilon > 0$, we can find a partition P such that for each $i \leq 4$,

$$|\int_a^b xf(x)dx - A_i| < \varepsilon.$$

This is clear for $i = 2, 4$, because for any P, A_2 (resp. A_4) is $L(g, P)$ (resp. upper sum $U(g, P)$), where g is the function $x \mapsto xf(x)$. But then if we make the partition very fine, making each $t_j - t_{j-1}$ very small, we can make $A_1(P) - A_2(P)$ and $A_3(P) - A_4(P)$ as small as we want. The claim follows.

Consequently, for every $\varepsilon > 0$, there exists P such that

$$|2\int_a^b xf(x)dx - A_1(P) + A_2(P)| < 2\varepsilon \tag{5.4.4}$$

and

$$|2\int_a^b xf(x)dx - A_1(P) + A_2(P)| < 2\varepsilon.$$

The assertion of the Proposition now follows by combining (5.4.3) and (5.4.4).

□

For a worked example, see problem (E) Assignment 5, for which a solution set has been posted on the web.

5.5 The integral test for infinite series

When we discussed the question of convergence of infinite series in chapter 2, we gave various tests one could use for this purpose, at least for series with non-negative coefficients. Here is another test, which can at times be helpful.

Proposition 5.5.1 *Consider an infinite series*

$$S = \sum_{n=1}^{\infty} a_n,$$

whose coefficients satisfy

$$a_n = f(n),$$

for some non-negative, monotone decreasing function f on the infinite interval $[1, \infty)$. Then S converges iff the improper integral

$$I = \int_1^\infty f(x)dx$$

converges.

Proof. For any integer $N > 1$, consider the partition

$$P_N : 1 < 2 < \ldots < N$$

of the closed interval $[0, N]$. Then, since f is monotone decreasing, the upper and lower sums are given by

$$U(f, P_N) = a_1 + a_2 + \ldots + a_{N-1}$$

and

$$L(f, P_N) = a_2 + \ldots + a_{N-1} + a_N.$$

Suppose f is integrable over $[1, \infty)$. Then it is integrable over $[1, N]$ and

$$a_1 + a_2 + \ldots + a_{N-1} \leq \int_1^N f(x)dx \leq a_2 + \ldots + a_{N-1} + a_N.$$

As N goes to infinity, this gives

$$S \leq \int_1^\infty f(x)dx,$$

which implies that S is convergent.

To prove the converse we need to be a bit more wily. Suppose S converges. Note that f is integrable over $[1, \infty)$ iff the series

$$T = \sum_{n=1}^\infty b_n$$

converges, where

$$b_n = \int_n^{n+1} f(x)dx$$

But since f is monotone decreasing over each interval $[n, n+1]$, the area under the graph of f is bounded above (resp. below) by the area under the constant function $x \mapsto f(n) = a_n$ (resp. $x \mapsto f(n+1) = a_{n+1}$). Thus we have, for every $n \geq 1$,

$$a_{n+1} \leq b_n \leq a_n.$$

Summing from $n = 1$ to ∞ and using the comparison test (see chapter 2), we get

$$S - a_1 \le T \le S.$$

Thus T converges as well.

$\square$

As a consequence we see that for any positive real number t, the series

$$S_t = \sum_{n=1}^{\infty} \frac{1}{n^t}$$

converges iff the improper integral

$$I_t = \int_1^{\infty} x^{-t} dx$$

converges. We have already seen that I_t is convergent iff $t > 1$. So the same holds for S_t. But recall that in the special case $t = 1$,we deduced the divergence of I_1 from that of S_1.