Notes on Calculus

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Fall 2001

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3 Basics of Integration

In this chapter we will study integration of reasonable functions over nice subsets A of $\mathbb{R}$. We will not consider here the integration of complex-valued functions or the integration over subsets of $\mathbb{C}$; that is in essence a part of *vector calculus* (Ma1c material), since $\mathbb{C}$ identifies with the *plane* $\{(x, y) \,|\, x, y \in \mathbb{R}\}$.

Before beginning our rigorous treatment of integration, which might be a bit abstract for some, here is the intuitive idea behind it. Given any function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$, we can consider its **graph** defined to be

$$\Gamma(f) \;=\; \{(x, y) \,|\, x \in A, y = f(x)\},$$

which is typically a curve in the plane. If f is non-negative on A, the graph will lie above the (horizontal) x-axis, though it might touch the x-axis at places. We would like to define the **integral of f over** A to be the **area**, if it makes sense, of the region between the graph of f and the part of the x-axis lying in A. It is helpful to first think of A as being a closed interval $[a, b]$, i.e, the set of points x in $\mathbb{R}$ with $a \leq x \leq b$. (Necessarily, $a < b$.) Then the area we want will be bounded by the graph Γ_f, the x-axis, and the vertical lines $\{y = f(a)\}$ and $\{y = f(b)\}$.

Consider, for example, the squaring function $f(x) = x^2$, with A being the interval $[0, 1]$. (See the figure enclosed as an attachment to this chapter.) How can we define and compute $I(f) = \int_0^1 f(x)dx$? For this we follow a method introduced by the famous nineteenth century German mathematician named Riemann. First note that $f(x)$ is a increasing function on this interval. Roughly speaking (see section 3.2 for a precise description), one chooses numbers $0 < t_1 << t_2 < \ldots < t_n = 1$ and looks at the *upper sum*, which corresponds in the attached figure to the area of the whole hatched area, and the *lower sum*, which corresponds within it, to the area of the darker region. One sees that the desired integral is caught between the upper and lower sums. As n gets larger, the difference Δ_n between the upper and lower sums gets smaller,which means that the hatched areas approximate the integral better. In the limit, as n goes to infinity, this difference becomes zero. In other words, the integral is the limit, as n goes to infinity, of either the upper sum sequence, or the lower sum sequence. For a general function $f(x)$, Δ_n may not converge to zero as n goes to infinity, and in such a case we will say that the integral does not exist. It is important to know that the integral might not exist in a given situation, even though in many practical situations f is continuous, and the integral exists.

3.1 Open, closed and compact sets in $\mathbb{R}$

Given any pair of real numbers a, b with $a < b$, one sets

$$(a, b) \;=\; \{x \in \mathbb{R} \,|\, a < x < b\},$$

called the **open interval** with endpoints a, b,

$$[a, b] \;=\; \{x \in \mathbb{R} \,|\, a \leq x \leq b\},$$

called the **closed interval** with endpoints a, b,

$$(a, b] = \{x \in \mathbb{R} \,|\, a < x \le b\},$$

and

$$[a, b) = \{x \in \mathbb{R} \,|\, a \le x < b\}.$$

The sets $(a, b]$ and $[a, b)$ are called **half-open** (or **half-closed**) intervals.

For $a \in \mathbb{R}$, we will call an interval of the form $(a - r, a + r)$ (for some $r > 0$) an **open interval centered at** a. Some people call it a **basic open set** in $\mathbb{R}$.

Remark. It is important to note that given any pair of open intervals I_1, I_2, we can find an open interval I contained in their intersection $I_1 \cap I_2$.

Given any subset X of $\mathbb{R}$, let us denote by X^c the complement $\mathbb{R} - X$ in $\mathbb{R}$. Clearly, the complement of the empty set $\emptyset$ is all of $\mathbb{R}$.

Let A be a subset of $\mathbb{R}$ and let y be a point in $\mathbb{R}$. Then there are exactly three possibilities for y relative to A:
(3.1.1)

(IP) There exists an open interval I containing y such that $I \subset A$.

(EP) There exists an open interval I centered at y which lies completely in the complement A^c of A.

(BP) Every open interval centered at y meets both A and A^c.

In case (IP), y is called an **interior point of** A. In case (EP), y is called an **exterior point of** A. In case (BP), y is called a **boundary point of** A. Note that in case (IP) $y \in A$, in case (EP) $y \notin A$, and in case (BP) y may or may not belong to A.)

Definition 3.1.2 *A set A in $\mathbb{R}$ is open if and only if every point of A is an interior point.*

Explicitly, this says: "Given any $z \in A$, we can find an open interval I containing z such that $I \subset A$."

Definition 3.1.3 A subset A of $\mathbb{R}$ is closed if its complement is open.

Lemma 3.1.4 *$A \subset \mathbb{R}$ is closed iff it contains all of its boundary points.*

Proof. Let y be a boundary point of A. Suppose y is not in A. Then it belongs to A^c, which is open. So, by the definition of an open set, we can find an open interval I containing y with $I \subset A^c$. Such a I does not meet A, contradicting the condition (BP). So A must contain y.

Conversely, suppose A contains all of its boundary points, and consider any z in A^c. Then z has to be an interior point or a boundary point of A^c. But the latter possibility does not arise as then z would also be a boundary point of A and hence belong to A (by hypothesis). So z is an interior point of A^c. Consequently, A^c is open, as was to be shown.

□

Examples:
(3.1.5)
(1) The empty set ϕ and $\mathbb{R}$ are both open and closed.

Since they are complements of each other, it suffices to check that they are both open, which is clear from the definition.

(2) Let $\{W_\alpha\}$ be a (possibly infinite and perhaps uncountable) collection of open sets in $\mathbb{R}$. Then their union $W = \cup_\alpha W_\alpha$ is also open.

Indeed, let $y \in W$. Then $y \in W_\alpha$ for some index α, and since W_α is open, there is an open set $V \subset W_\alpha$ containing y. Then we are done as $y \in V \subset W_\alpha \subset W$.

(3) Let $\{W_1, W_2, \cdots, W_n\}$ be a **finite** collection of open sets. Then their intersection $W = \bigcap_{j=1}^{n} W_j$ is open.

Proof. Let $y \in W$. Then $y \in W_j$, $\forall j$. Since each W_j is open, we can find an open interval I_j such that $y \in I_j \subset W_j$. Then, by the remark above, we can find an open interval I contained in the intersection of the I_j such that $y \in I$. Done.

Warning: The intersection of an infinite collection of open sets need not be open, as shown by the following (counter)example. Put, for each $k \geq 1$, $W_k = \left(-\frac{1}{k}, \frac{1}{k}\right)$. Then $\cap_k W_k = \{0\}$, which is not open.

(4) Any **finite** set of points $A = \{P_1, \ldots, P_r\}$ is closed.
Proof. For each j, let U_j denote the complement of P_j (in $\mathbb{R}$). Given any z in U_j, we can easily find an open interval V_j containing z which avoids P_j. So U_j is open, for each j. The complement of A is simply $\cap_{j=1}^{r} U_j$, which is then open by (4).

More generally, one can show, by essentially the same argument, that a *finite* union of closed sets is again closed.

It is important to remember that there are many sets A in $\mathbb{R}$ which are neither open nor closed. For example, look at the half-closed, half-open interval $[0, 1)$ in $\mathbb{R}$.

It is customary to extend slightly our notion of an open interval, which some would call a **finite open interval**, and define **infinite open intervals**, for all $a \in \mathbb{R}$, by

$$(a, \infty) \;=\; \{x \in \mathbb{R} \,|\, a < x\}$$

and

$$(-\infty, a) \;=\; \{x \in \mathbb{R} \,|\, x < a\}.$$

At times we will also write $(-\infty, \infty)$ for $\mathbb{R}$.

We are not (at all) claiming here that there are real numbers called ∞ and $-\infty$. They are introduced mainly for notational convenience. It is seen easily that these infinite open intervals are indeed open – use example (2) above.

One also defines **infinite half-open intervals** (for all $a \in \mathbb{R}$) by

$$[a, \infty) \;=\; \{x \in \mathbb{R} \,|\, a \leq x\},$$

and

$$(-\infty, a] \,=\, \{x \in \mathbb{R} \,|\, x \leq a\}.$$

Since there are no such numbers as ∞ and $-\infty$ in the real number system, it does not make sense to define closed infinite intervals.

We will call a subset Y of $\mathbb{R}$ a **bounded set** iff we can enclose it in a closed interval, i.w., $Y \subset [a, b]$ for some real numbers a, b with $a < b$.

This clearly agrees with the notion of boundedness of sequences encountered in the previous chapter.

Definition 3.1.6 *A subset C of $\mathbb{R}$ which is both closed and bounded is called a* **compact set***.*

This definition implies, in particular, that **any closed interval** $[a, b]$ **is compact**. It will turn out later on that compact sets get sent to compact sets under continuous functions.

3.2 Integrals of bounded functions

We will first discuss the question of integrability of bounded functions on closed intervals, and then move on to integration of continuous functions, and then to integration over more general sets in $\mathbb{R}$. The main tool will be to approximate the integral from above by the **upper sum** and from below by the **lower sum**, relative to various partitions. This method was introduced by the famous nineteenth century German mathematician Riemann, and it is customary to call these sums **Riemann sums**.

To begin, let us define the **length** of a closed interval $[a, b]$ to be

$$\ell([a, b]) \,=\, b - a.$$

Note that the *interior* of $[a, b]$ is simply the open interval (a, b).

By a **bounded function** on a set A, we will mean a function

$$f \,:\, A \,\to\, \mathbb{R},$$

such that the *image* $f(A)$ of f is bounded.

Definition 3.2.1 *A* **partition** *of a closed interval $[a, b]$ is a finite collection P of closed subintervals $J_1, J_2, \ldots, J_r \subseteq [a, b]$ such that*

(i) $[a, b] = \cup_{j=1}^{r} J_j$, and

(ii) the interiors of J_i and J_j have no intersection if $i \neq j$.

Giving such a partition of $[a, b]$ is evidently the same as giving $r+1$ points $t_0, t_1, \ldots, t_{r-1}, t_r$ in $\mathbb{R}$ such that

(3.2.2) $$a = t_0 \,<\, t_1 \,<<\, \ldots \,<\, t_{r-1} \,<\, t_r = b.$$

The corresponding intervals J_j are obtained by setting

$$J_j = [t_{j-1}, t_j] \quad \forall j \geq 1.$$

Definition 3.2.3 *A* **refinement** *of a partition $P : a = t_o < t_1 < \ldots < t_n = b$ of $[a, b]$ is another partition $P' : a = t'_0 < t'_1 < \ldots < t'_m = b$ with $\{t_i | 1 \leq i \leq n\}$ contained in the set $\{t'_j | 1 \leq j \leq m\}$.*

It is clear from the definition that given any two partitions P, P' of $[a, b]$, we can find a third partition P'' which is simultaneously a refinement of P and of P'. Indeed, if P is given by the set $\{t_i \,|\, 0 \leq i \leq r\}$ and P' by $\{u_j \,|\, 0 \leq j \leq s\}$, P'' can be taken to be defined by the union $\{t_i, u_j \,|\, 0 \leq ir, 0 \leq j \leq s\}$. Such a P'' is called a **common refinement** of P, P'.

Now let f be a bounded function on $[a, b]$, and let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ a partition of $[a, b]$. Then f is certainly bounded on each $[T_{j-1}, t_j]$. Remember that it was proved in Chapter 1 that every bounded subset of $\mathbb{R}$ admits a **sup**, the **least upper bound**, and an **inf**, the **greatest lower bound**.

Definition 3.2.4 *The* **upper**, *resp.* **lower**, **sum** *of f over $[a, b]$ relative to the partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ is given by*

$$U(f, P) = \sum_{j=1}^{r} (t_j - t_{j-1}) \sup(f([T_{j-1}, t_j]))$$

resp.

$$L(f, P) = \sum_{j=1}^{r} (t_j - t_{j-1}) \inf(f([T_{j-1}, t_j])).$$

Of course we have

$$L(f, P) \leq U(f, P), \qquad \text{for all } P.$$

More importantly, it is clear from the definition that if $P' = \{J'_k\}_{k=1}^{m}$ is a refinement of P, then

$$L(f, P) \leq L(f, P') \quad \text{and} \quad U(f, P') \leq U(f, P).$$

Put

$$\mathcal{L}(f) = \{L(f, P) \mid P \text{ partition of } [a, b]\} \subseteq \mathbb{R} \tag{3.2.5}$$

and

$$\mathcal{U}(f) = \{U(f, P) \mid P \text{ partition of } [a, b]\} \subseteq \mathbb{R}.$$

Lemma 3.2.6 *$\mathcal{L}(f)$ admits a* **sup**, *denoted $\underline{I}(f)$, and $\mathcal{U}(f)$ admits an* **inf**, *denoted $\overline{I}(f)$.*

Proof. Thanks to the discussion in Chapter 1, all we have to do is show that $\mathcal{L}(f)$ (resp. $\mathcal{U}(f)$) is bounded from above (resp. below). So we will be done if we show that given any two partitions P, P' of $[a, b]$, we have $L(f, P) \leq U(f, P')$ as then $\mathcal{L}(f)$ will have $U(f, P')$ as an upper bound and $\mathcal{U}(f)$ will have $L(f, P)$ as a lower bound. Choose a third partition P'' which refines P and P' simultaneously. Then we have $L(f,P) \le L(f,P'') \le U(f,P'') \le U(f,P')$. Done.

$\square$

We always have
$$\underline{I}(f) \le \overline{I}(f).$$
It is customary to call $\underline{I}(f)$ the **lower integral**, and $\overline{I}(f)$ the **upper integral**, of f over $[a,b]$.

Definition 3.2.7 *A bounded function f on $[a,b]$ is* **integrable** *iff $\underline{I}(f) = \overline{I}(f)$. When such an equality holds, we will simply write $I(f)$ (or $I_{[a,b]}(f)$ if the dependence on $[a,b]$ needs to be stressed) for $\underline{I}(f)$ $(= \overline{I}(f))$, and call it the* **integral of f over** $[a,b]$.

Quite often we will write
$$I(f) = \int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx.$$

In practice one is loathe to consider *all* partitions P of $[a,b]$. The following lemma tells us something useful in this regard.

Lemma 3.2.8 *Let f be a bounded function on $[a,b]$. Suppose $\{P_n\}$ is an infinite sequence of partitions, with each P_n being a refinement of P_{n-1}, such that the corresponding sequences $\{U(f,P_n\}$ and $\{L(f,P_n)\}$ both converge to a common limit λ in $\mathbb{R}$. Then f is integrable with $\lambda = \int_a^b f(x)dx$.*

The obvious question now is to ask if there are integrable functions. One such example is given by the **constant function** $f(x) = c$, for all $x \in [a,b]$. Then for any partition $P = \{A = t_0 < t_1 < \ldots < t_n = b\}$, we have
$$L(f,P) = U(f,P) = c\sum_{j=1}^{r}(t_j - t_{j-1}) = c(b-a).$$
So $\underline{I}(f) = \overline{I}(f)$ and
$$\int_a^b f = c(b-a).$$

This can be jazzed up as follows.

Definition *A* **step function on** $[a,b]$ *is a function f on $[a,b]$ which is constant on each of the closed subintervals $[t_{j-1}, t_j]$ of* **some** *partition P.*

Lemma 3.2.9 *Every step function f on $[a,b]$ is integrable.*

Proof. By definition, there exists a partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a, b]$ and scalars $\{c_j\}$ such that $f(x) = c_j$, if $x \in [T_{j-1}, t_j]$. Then, arguing as above, it is clear that for **any** refinement P' of P, we have

$$L(f, P') = U(f, P') = \sum_{j=1}^{n} c_j \,(t_j - t_{j-1}).$$

Hence, $\underline{I}(f) = \overline{I}(f)$.

□

Note that for such a step function f defined by $(P, \{c_j\})$, we have an explicit formula for the integral, namely

$$\int\limits_a^b f(x)dx \;=\; \sum_{j=1}^{n} c_j \,(t_j - t_{j-1}).$$

3.3 Integrability of monotone functions

Let $f : A \to \mathbb{R}$ be a function, with A a subset of $\mathbb{R}$. We will call f **monotone increasing**, resp. **monotone decreasing**, iff we have

$$x_1, x_2 \in A,\, x_1 < x_2 \implies f(x_1) \le f(x_2) \quad (\text{resp.} \quad f(x_1) \ge f(x_2)).$$

By a **monotone** function, we will mean a function which is either monotone increasing or monotone decreasing.

Lemma 3.3.1 *Let f be a monotone function on a closed interval $[a, b]$. Then f is bounded on $[a, b]$.*

Proof. Suppose f is monotone increasing. Then by definition, $f(a) \le f(x) \le f(b)$ for all x in $[a, b]$. Hence f is bounded by $f(a)$ (from below) and $f(b)$ (from above). If f is monotone decreasing, the proof is similar and will be left as an exercise.

□

Theorem 3.3.2 *Let f be a monotone function on $[a, b]$. Then f is integrable.*

Proof. Suppose f is monotone increasing on $[a, b]$. For $n \ge 1$, let $P_n = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be the partition with

$$t_{j+1} - t_j \;=\; \frac{b-a}{n} \quad \forall\, j \le n - 1.$$

Since f is increasing, $f(t_j)$ is, for each j, the *inf* of the values of f on the subinterval $[t_j, t_{j+1}]$, and $f(t_{j+1})$ is the *sup*. Hence

$$L(f, P_n) \;=\; \frac{b-a}{n}\,(f(t_0) + f(t_1) + \ldots + f(t_{n-1}))$$

and

$$U(f, P_n) = \frac{b-a}{n} \left(f(t_1) + f(t_2) + \ldots + f(t_n) \right).$$

It follows that

$$U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \left(f(b) - f(a) \right).$$

As n goes to infinity, this difference goes to zero. Hence f is integrable by Lemma 3.2.8.

□

3.4 Computation of $\int_a^b x^s dx$

The function $f(x) = x^m$ is a monotone increasing function for any $m > 0$. So by Theorem 3.3.2, it is integrable on any closed interval $[a, b]$. There is a well known formula for the value of the integral, which can be derived in a myriad of ways. We will give two proofs in this chapter, and here is the first one – due to Riemann, which uses partitions $P = a = t_0 < t_1 < \ldots < t_n = b$ where the subintervals $[t_i, t_{i+1}]$ are not of equal length, but where the ratios t_{i+1}/t_i are kept constant.

Riemann's method, unlike the one we will describe later in section 3.7, is very general, and works also for x^s for *any* real exponent $s \neq -1$, as long as the limits a, b are positive. We have encountered x^s before for rational s, and for those who know about logarithms and exponentials, x^s is defined for any real s and positive x as $e^{s \log x}$; here $\log x$ denotes the *natural logarithm* of x, which some denote by $ln(x)$. It is problematic to define x^s for general s and negative x, except when s is an integer, because numbers like $(-1)^{1/2}$ are not in $\mathbb{R}$.

Proposition 3.4.1 *Let s, a, b be real numbers with $s \neq -1$ and $a < b$. If s is not an integer, assume that a is positive. Then*

$$\int_a^b x^s dx = \frac{b^{s+1} - a^{s+1}}{s+1}.$$

We will prove this result below only for positive integer values of s, but we will make a remark after the proof about what one needs for the extension to general s.

When $s = -1$, one cannot divide by $s + 1$ and the Proposition cannot hold as stated. It may be useful to note that for any $b > 1$,

$$\int_1^b \frac{1}{x} dx = \log b.$$

One can take this to be the definition of $\log b$.

Proof of Proposition for positive integral exponents.

We will take s to be a positive integer m. (The assertion is obvious for $m = 0$.) We will also assume, for simplicity of exposition, that $a > 0$, even though the asserted formula holds equally well when $a \leq 0$. Write

$$f(x) = x^m.$$

Put

$$u = b/a > 1, \tag{3.4.2}$$

and define, for each $n \geq 1$, a partition

$$P_n : a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b \tag{3.4.3}$$

such that for each $j \leq n$,

$$t_j = au^{j/n}.$$

Then we have, for all $j < n$,

$$\frac{t_{j+1}}{t_j} = u^{(j+1)/n - j/n} = u^{1/n}, \tag{3.4.4}$$

which is independent of j.

The lower sum is given, for each n, by

$$L(f, P_n) = \sum_{j=0}^{n-1} (t_{j+1} - t_j) t_j^m = (u^{1/n} - 1) \sum_{j=0}^{n-1} t_j^{m+1}, \tag{3.4.5}$$

where we have used (3.4.4). By the definition of t_j and the fact that $\sum_{j=0}^{n-1} z^j = \frac{1-z^n}{1-z}$, the expression on the right becomes

$$a^{m+1}(u^{1/n} - 1) \sum_{j=0}^{n-1} u^{j(m+1)/n} = -a^{m+1} \frac{(1 - u^{1/n})(1 - u^{m+1})}{1 - u^{(m+1)/n}}.$$

Since $-a^{m+1}(1 - u^{m+1})$ equals $b^{m+1} - a^{m+1}$, we get

$$L(f, P_n) = (b^{m+1} - a^{m+1}) \frac{1 - u^{1/n}}{1 - u^{(m+1)/n}} = (b^{m+1} - a^{m+1}) \frac{1}{1 + u^{1/n} + u^{2/n} + \ldots + u^{m/n}}. \tag{3.4.6}$$

We know that as $n \to \infty$, $u^{j/n}$ goes to 1 for any fixed j. This shows that

$$\lim_{n \to \infty} \frac{1}{1 + u^{1/n} + u^{2/n} + \ldots + u^{m/n}} = \frac{1}{m+1}. \tag{3.4.7}$$

Consequently, by (3.4.6),

$$\lim_{n \to \infty} L(f, P_n) = \frac{b^{m+1} - a^{m+1}}{m+1}. \tag{3.4.8}$$

On the other hand, since $t_{j+1}^m = u^{m/n}t_j^m$, the corresponding upper sum is

$$U(f,P_n) \;=\; \sum_{j=0}^{n-1}(t_{j+1}-t_j)(u^{m/n}t_j^m) \;=\; u^{m/n}L(f,P_n). \tag{3.4.9}$$

And since $u^{m/n} \to 1$ as $n \to \infty$, (3.4.8) and (3.4.9) imply that we have

$$\lim_{n\to\infty} U(f,P_n) \;=\; \lim_{n\to\infty} L(f,P_n) \;=\; \frac{b^{m+1}-a^{m+1}}{m+1}.$$

It follows then (see Lemma 3.2.8) that the (definite) integral of x^m over $[a,b]$ equals this common limit. Incidentally, this computation shows explicitly that $f(x) = x^m$ is integrable and we don't really need to refer to Theorem 3.3.2.

□

Now suppose we want to prove the full force of the Proposition, i.e., treat the case of an arbitrary real exponent $s \neq -1$, by this method. Proceeding as above, we will get

$$L(f,P_n) \;=\; (b^{s+1}-a^{s+1})\phi_s(n), \tag{3.4.6}$$

where

$$\phi_s(n) \;=\; \frac{1-u^{1/n}}{1-u^{(s+1)/n}},$$

and

$$U(f,P_n) \;=\; u^{s/n}L(f,P_n).$$

As before, $u^{s/n}$ goes to 1 as $n \to \infty$. So the whole argument will go through if we can establish the following limit:

$$\lim_{n\to\infty} \phi_s(n) \;=\; \frac{1}{s+1}.$$

This can be done, but we will not do it here. In any case, the students should feel free to use the Proposition for all $s \neq -1$.

3.5 Example of a non-integrable, bounded function

Define a function

$$f : [0,1] \to \mathbb{R} \tag{3.5.1}$$

by the following recipe. If x is irrational, set $f(x) = 0$, and if x is rational, put $f(x) = 1$.

This certainly defines a bounded function on [0.1], and one is led to wonder about the integrability of f.

Proposition 3.5.2 *This f is not integrable.*

Proof. Let P be any partition of $[0,1]$ given by $0 = t_0 < t_1 < \ldots < t_n = 1$. By the property $\mathcal{R}4$ of $\mathbb{R}$ (see section 1.1), we know that there is a rational number between any two real numbers. So in every subinterval $[t_j, t_{j+1}]$ there will be some rational number q_j (in fact infinitely many), with $f(q_j) = 1$ by definition. Consequently,

$$U(f, P) \;=\; \sum_{j=0}^{n-1} 1 \cdot (t_{j+1} - t_j) \;=\; (t_0 - t_1) + (t_1 - t_2) + \ldots + (t_{n-1} - t_n) \;=\; 1,$$

because $t_0 = 0$ and $t_n = 1$.

On the other hand, every interval $[c, d]$ in $\mathbb{R}$ must contain an irrational number y. Let us give a proof. If c or d is irrational, then we may take y to be that number, so we can assume that c and d are rational. Then the number $y = c + (d - c)\sqrt{2}/2$ is irrational and lies in $[c, d]$. Consequently, for every j, there is an irrational y_j in $[t_j, t_{j+1}]$, which implies that

$$L(f, P) \;=\; 0.$$

Hence

$$U(f, P) - L(f, P) \;=\; 1,$$

and this is independent of the partition P. So f is not integrable.

□

It should be noted, however, that there are non-zero integrable, bounded functions f on $[0, 1]$ which are supported on $\mathbb{Q}$, i.e., $f(x)$ is zero for irrational numbers x. But they are not constant on the rational numbers.

Here is a very useful simple fact about rational numbers in the interval $[0, 1]$. For every positive integer n, there are at most n rational numbers whose reduced fraction expressions have n as the denominator. This is also the fact that allows one to establish a one-to-one correspondence between $\mathbb{Q}$ and $\mathbb{N}$, showing that $\mathbb{Q}$ is countable.

3.6 Properties of integrals

The integral $\int_a^b f(x)dx$ is often called a **definite integral** because it has a definite value, assuming that f is integrable. One calls f the **integrand**, a the **lower limit** and b the **upper limit**. It is customary to use the convention

$$\int_b^a f(x)dx \;=\; -\int_a^b f(x)dx. \tag{3.6.0}$$

The definite integral has many nice properties as f or $[a, b]$ varies, making our life very pleasant, which we want to discuss in this section.

Proposition 3.6.1 (Linearity in the integrand) *If f, g are integrable over $[a, b]$, so is any linear combination $\alpha f + \beta g$, with $\alpha, \beta \in \mathbb{R}$, and moreover,*

$$\int_a^b \{\alpha f(x) + \beta g(x)\}dx \;=\; \alpha \int_a^b f(x) + \beta \int_a^b g(x)dx.$$

Proof. If P is any partition, it is immediate from the definition that

$$L(\alpha f + \beta g, P) \;=\; \alpha L(f, P) + \beta L(g, P)$$

and

$$U(\alpha f + \beta g, P) \;=\; \alpha U(f, P) + \beta U(g, P).$$

It follows then that

$$\underline{I}(\alpha f + \beta g) \;=\; \alpha \underline{I}(f) + \beta \underline{I}(g)$$

and

$$\overline{I}(\alpha f + \beta g) \;=\; \alpha \overline{I}(f) + \beta \overline{I}(g).$$

Since f, g are integrable, $\underline{I}(f) = \overline{I}(f)$ and $\underline{I}(g) = \overline{I}(g)$. So the lower and upper integrals of $\alpha f + \beta g$ coincide, proving the assertion.

$\square$

Proposition 3.6.2 (Additivity in the limits) Let a, b, c are real numbers with $a < b < c$, and let f be integrable on $[a, c]$. Then f is integrable on $[a, b]$ and $[b, c]$, and moreover,

$$\int_a^c f(x)dx \;=\; \int_a^b f(x)dx \;+\; \int_b^c f(x)dx.$$

Proof. Given partitions P, P' of $[a, b], [c, d]$ respectively, $P \cup P'$ defines a partition of $[a, c]$. And if P'' is a partition of $[a, c]$, then we can refine it by adding b to get a partition of the type $P \cup P'$, with P (resp. P') being a partition of $[a, b]$ (resp. $[b, c]$). It follows easily that the lower (resp. upper) integral of f over $[a, c]$ is the sum of the lower (resp. upper) integrals of f over $[a, b]$ and $[b, c]$. The assertion follows.

$\square$

If c is any real number , the function $x \to x + c$ is called **translation** by c. The following Proposition describes the **translation invariance** of the definite integral.

Proposition 3.6.3 *Suppose f is integrable on $[a, b]$ and $c \in \mathbb{R}$. Then the c-**translate** of f, given by $x \to f(x + c)$, is integrable on $[a - c, b - c]$, and*

$$\int_{a-c}^{b-c} f(x + c)dx \;=\; \int_a^b f(x)dx.$$

For any $c \in \mathbb{R}$, the function $x \to cx$ is called the **homothety** (or **stretching**) by c. Some also use the terms **expansion** and **contraction** when $c > 1$ and $0 < c < 1$. The following Proposition describes the **behavior under homothety**.

Proposition 3.6.4 *Suppose f integrable on $[ac, bc]$. Then the function $x \to f(cx)$ is integrable on $[a, b]$ and*

$$\int_{ac}^{bc} f(x)dx \;=\; c \int_a^b f(cx)dx.$$

3.7 The integral of x^m revisited, and polynomials

In Proposition 3.4.1 we established the following identity for any $m \geq 0$ and any $[a, b]$:

$$(*) \qquad \int_a^b x^m dx = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

In view of the linearity property of the integral (see Proposition 3.6.1), we see that if we have any **polynomial**

$$f(x) = c_0 + c_1 x + \ldots + c_n x^n,$$

then f is integrable over any $[a, b]$. Furthermore, we have the **explicit formula**

$$(**) \qquad \int_a^b f(x)dx = c_0 x + c_1 \frac{x^2}{2} + \ldots + c_n \frac{x^{n+1}}{n+1}.$$

We will now give an **alternate proof** of $(*)$ for $0 < a < b$.

We first need a Lemma, which is of independent interest. We will call a function $f(x)$ **even**, resp. **odd**, iff $f(-x) = f(x)$, resp. $f(-x) = -f(x)$, for all x. Note that x^j is even if j is even and it is odd if j is odd. Also, $\cos x$ is even while $\sin x$ is odd.

Lemma 3.7.1 *Let $f(x)$ be an integrable function of $[-a, a]$, for some $a > 0$. Then*

$$f \text{ even} \quad \Longrightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx,$$

$$f \text{ odd} \quad \Longrightarrow \int_{-a}^{a} f(x)dx = 0.$$

Proof. By Proposition 3.6.4 (and convention (3.6.0)),

$$\int_{-a}^{0} f(x)dx = \int_0^a f(-x)dx,$$

which equals

$$(-1)^r \int_0^a f(x)dx,$$

with r being 1, resp. -1, when f is even, resp. odd. The lemma now follows by Proposition 3.6.2, which implies that

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx.$$

$\square$

Now let us begin the alternate proof of $(*)$. Put

$$I_m = \int_0^1 x^m dx.$$

The key is to show that

$$I_m = \frac{1}{m+1}. \tag{3.7.2}$$

Indeed, we can use Proposition 3.6.4 to deduce that

$$\int_0^b x^m dx = b \int_0^1 (bx)^m dx = b^{m+1} I_m, \tag{3.7.3}$$

and combining this with Proposition 3.6.2, we get

$$\int_a^b x^m dx = \int_0^b x^m dx - \int_0^a x^m dx = (b^{m+1} - a^{m+1}) I_m, \tag{3.7.4}$$

as desired. So let us now prove (3.7.2).

By the translation invariance (Proposition 3.6.3) of definite integrals, we get

$$\int_{-b}^b (x+b)^m dx = \int_0^{2b} x^m dx = 2^{m+1} b^{m+1} I_m. \tag{3.7.5}$$

By the binomial theorem,

$$(x+b)^m = \sum_{j=0}^m \binom{m}{j} x^j b^{m-j}.$$

and so by (3.7.5),

$$2^{m+1} b^{m+1} I_m = \sum_{j=0}^m \binom{m}{j} b^{m-j} \int_{-b}^b x^j dx. \tag{3.7.6}$$

By Lemma 3.7.1, $\int_{-b}^b x^j dx$ equals 0 if j is odd, and twice $\int_0^b x^j dx = 2b^{j+1} I_j$ (by 3.7.3) when j is even.

Since $(*)$ is true for $m = 0$, we can take $m > 0$ and assume, by induction, that $(*)$ holds for all $j < m$. Then we get from the above, the following:

$$2^m I_m = \sum_{j=0,\, j\,\mathrm{even}}^{m} \binom{m}{j} I_j = \epsilon_m I_m + \sum_{j=0,\, j\,\mathrm{even}}^{m-1} \binom{m}{j} \frac{1}{j+1}, \tag{3.7.7}$$

where ϵ_m is 1 if m is even and 0 if m is odd. On the other hand,

$$\binom{m}{j}\frac{1}{j+1} = \frac{m!}{(j+1)!(m-j)!} == \binom{m+1}{j+1}\frac{1}{m+1},$$

so that

$$\sum_{j=0,\, j\,\mathrm{even}}^{m-1} \binom{m}{j}\frac{1}{j+1} = \frac{1}{m+1} \sum_{k=0,\, k\,\mathrm{odd}}^{m} \binom{m+1}{k}. \tag{3.7.8}$$

Next we note that for any integer $r \geq 1$,

$$\sum_{k=0,\, k\,\mathrm{odd}}^{r} \binom{r}{k} = \frac{1}{2}(1-(-1))^r = 2^{r-1}.$$

Consequently,

$$\sum_{k=0,\, k\,\mathrm{odd}}^{m} \binom{m+1}{k} = 2^m - \epsilon_m. \tag{3.7.9}$$

Combining (3.7.7.), (3.7.8) and (3.7.9), we get

$$2^m I_m = \epsilon_m I_m + \frac{2^m}{m+1} - \epsilon_m \frac{1}{m+1}.$$

When m is odd, $\epsilon_m = 0$ and hence

$$2^m I_m = \frac{2^m}{m+1}.$$

When m is even, $\epsilon_m = 1$ and we get

$$2^m I_m = I_m + \frac{2^m}{m+1} - \frac{1}{m+1}.$$

In either case,

$$I_m = \frac{1}{m+1}$$

as asserted.

□