Notes on Calculus

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Contents

2 Sequences and Series

In this chapter we will study two related questions. Given an infinite collection X of numbers, which can be taken to be rational, real or complex, the first question is to know if there is a limit point to which the elements of X congregate (converge). The second question is to know if one can add up all the numbers in X.

Even though we will be mainly thinking of $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, many things will go through for numbers in any field K with an absolute value $|\,.\,|$ satisfying $|0| = 0$, $|x| > 0$ if $x \neq 0$, $|xy| = |x| \cdot |y|$, and the *triangle inequality* $|x + y| \leq |x| + |y|$.

A necessary thing to hold for convergence is the *Cauchy condition*, which is also sufficient if we work with $\mathbb{R}$ or $\mathbb{C}$, but not $\mathbb{Q}$. What we mean by this remark is that a Cauchy sequence of rational numbers will have a limit L in $\mathbb{R}$, but L need not be rational. One can enlarge $\mathbb{Q}$ to have this criterion work, i.e., adjoin to $\mathbb{Q}$ limits of Cauchy sequences of rational numbers, in which case one gets yet another construction of $\mathbb{R}$. This aspect is discussed in section 2.3, which one can skip at a first reading.

The last three sections deal with the second question, and we will derive various *tests* for *absolute convergence*. Sometimes, however, a series might converge though not absolutely. A basic example of this phenomenon is given by the *Leibniz series*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}.$$

But the series on the left is *not* absolutely convergent, i.e., the series $1 + \frac{1}{3} + \frac{1}{5} + \dots$ diverges.

2.1 Convergence of sequences

Let K denote either $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. When we say a *number*, or a *scalar*, we will mean an element of K. But by a *positive number* we will mean a positive *real* number. (One could restrict to positive rational numbers instead, and it will work just as well to use this subset.)

By a *sequence*, or more properly an *infinite sequence*, we will mean a collection of numbers

$$a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots,$$

which is indexed by the set $\mathbb{N}$ of natural numbers. We will often denote it simply as $\{a_n\}$.

A simple example to keep in mind is given by $a_n = \frac{1}{n}$, which appears to decrease towards zero as n gets larger and larger. In this case we would like to have 0 declared as the *limit of the sequence*.

It should be noted that if we are given a collection of numbers indexed by any countably infinite set, we can reindex it by $\mathbb{N}$ and apply all our results below to it. For example, given $M > 0$ and $\{c_n \,|\, n > -M\}$, we can renumber it by putting $a_n = c_{n+M}$ and get a sequence indexed by $\mathbb{N}$.

We will say that a sequence $\{a_n\}$ **converges to a limit** L in K iff for every positive number ϵ there exists an index N such that

(2.1.1) $$n > N \implies |L - a_n| < \epsilon.$$

It is *very important* to note that we have to check the condition above for *every* positive ϵ. Clearly, if (2.2.1) holds for N, it also holds for any $N' > N$.

When $\{a_n\}$ does converge to a limit L, we will write

$$a_n \to L \quad \text{or} \quad \lim_{n\to\infty} a_n = L.$$

When we write $n \to \infty$, we do not necessarily mean that there is a number called ∞, which we have certainly not defined; we mean only that n becomes arbitrarily large. In other words we want to know what happens to a_n when n is larger than any number one can think of.

Let us quickly prove that $\lim_{n\to\infty} \frac{1}{n}$ is 0, as expected. For any positive number ϵ, we can find a rational number N such that $0 < \frac{1}{N} < \epsilon$. Indeed, by the property $\mathcal{R}4$ of $\mathbb{R}$ (see section 1.0), we can find a rational number q with $0 < q < \epsilon$, and since q s a positive rational number, we can find an $N > 0$ such that $\frac{1}{N} < q$. So for every $n > N$, we have

$$|0 - \frac{1}{n}| = \frac{1}{n} < \frac{1}{N} < \epsilon,$$

proving that $\frac{1}{n}$ does converge to 0.

When a sequence does not converge to any number L, we will say that it *diverges*. A simple example is given by the sequence $\{1, -1, 1, -1, 1, -1, \dots\}$. Given any $\epsilon < 1$, we will be unable to find an N for which (2.1.1) holds for any L. This is so because for any N $\{a_{N+1}, a_{N+2}\} = \{1, -1\}$, and (for any L) at least one of them will be at a distance > 1 from L; to see this use the triangle inequality $|L - a_{N+1}| \leq |L - a_{N+2}| + |a_{N+2} - a_{N+1}|$, together with $|a_{N+2} - a_{N+1}| = 2$.

Any sequence which converges (to some limit) will be called a *convergent sequence*, and one that does not converge will be called a *divergent sequence*.

Proposition 2.1.2

(i) If $\{a_n\}$ is a convergent sequence with limit L, then for any scalar c, the sequence $\{ca_n\}$ is convergent with limit cL;

(ii) If $\{a_n\}$, $\{b_n\}$ are convergent sequences with respective limits L_1, L_2, then their sum $\{a_n + b_n\}$ and their product $\{a_n b_n\}$ are convergent with respective limits $L_1 + L_2$ and $L_1 L_2$.

The proof is a simple application of the properties of absolute values. To illustrate, let us prove the assertion about the sum sequence $\{a_n + b_n\}$. Pick any $\epsilon > 0$. Since $a_n \to L_1$ and $b_n \to L_2$, we can find $N > 0$ such that

$$n > N \implies |L_1 - a_n| < \frac{\epsilon}{2}, \; |L_2 - b_n| < \frac{\epsilon}{2}.$$

Using the triangle inequality we then have, for all $n > N$,

$$|(L_1 + L_2) - (a_n + b_n)| \leq |L_1 - a_n| + |L_2 - b_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $a_n + b_n \to L_1 + L_2$.

Corollary 2.1.3 *Let $\{z_n = x_n + iy_n\}$ be a sequence of complex numbers, with x_n, y_n real for each n. Then $\{z_n\}$ converges iff the real sequences $\{x_n\}$ and $\{y_n\}$ are both convergent.*

Proof. Suppose $\{x_n\}, \{y_n\}$ are both convergent, with respective limits u, v. We claim that $\{z_n\}$ then converges to $w = u+iv$. Indeed, by the Proposition above, $\{iy_n\}$ is convergent with limit iv, and so is $\{x_n + iy_n\}$, with limit w. Conversely, suppose that $\{z_n\}$ converges, say to w. We may write w as $u + iv$, with u, v real. For any complex number $z = x + iy$, $|x|$ and $|y|$ are both bounded by $\leq \sqrt{x^2 + y^2}$, i.e., by $|z|$. Since $w - z_n = (u - x_n) + i(v - y_n)$, we get

$$|u - x_n| \leq |w - z_n| \quad \text{and} \quad |v - y_n| \leq |w - z_n|.$$

For any $\epsilon > 0$, pick $N > 0$ such that for all $n > N$, $|w - z_n|$ is $< \epsilon$. Then we also have $|u - x_n| < \epsilon$ and $|v - y_n|$ is $< \epsilon$ for all $n > N$, establishing the convergence of $\{x_n\}$ and $\{y_n\}$ with respective limits u and v.

□

Thanks to this Corollary, the study of sequences of complex numbers can be reduced to that of real numbers.

A sequence $\{a_n\}$ is *bounded* iff there is a positive number M such that $|a_n| \leq M$ for all n.

Lemma 2.1.4 If a sequence $\{a_n\}$ converges, then it must be bounded.

Proof Let L be the limit of $\{a_n\}$. Fix any positive number ϵ. Then by definition, there exists an $N > 0$ such that for all $n > N$, $|L - a_n|$ is $< \epsilon$. By the triangle inequality,

$$|a_n| \leq |a_n - L| + |L|.$$

So if we set

$$M = \max\{a_1, a_2, \ldots, a_N, L + \epsilon\},$$

then every $|a_n|$ is bounded by M.

□

The converse of this Lemma is *not* true: Remember that the sequence $\{1, -1, 1, -1, \ldots\}$ has not limit, but it is bounded in absolute value by 1.

For any real number x and consider the *power sequence* $\{x^n\}$. When $x = 1$ (resp. 0), we get the *stationary sequence* $\{1, 1, 1, \ldots\}$ (resp. $\{0, 0, 0, \ldots\}$), which is obviously convergent with limit 1 (resp. 0). When $x = -1$, we get $\{-1, 1, -1, \ldots\}$, which is divergent, as we saw above. Applying the following Lemma we see that this sequence is unbounded when $|x| > 1$, hence divergent, and that it converges to zero when $|x| < 1$.

Lemma 2.1.5 *Let y be a positive real number. Then $\{y^n\}$ is unbounded if $y > 1$, while*

$$\lim_{n\to\infty} y^n = 0 \quad \text{if} \quad y < 1.$$

Proof of Lemma. Suppose $y > 1$. Write $y = 1 + t$ with $t > 0$. Then by the binomial theorem (which can be proved by induction),

$$y^n = (1+t)^n = \sum_{k=1}^{n} \binom{n}{k} t^k,$$

which is $\geq 1 + nt$. Since $1 + nt$ is unbounded, i.e, larger than any number for a big enough n, y^n is also unbounded.

Now let $y < 1$. Then y^{-1} is > 1 and hence $\{y^{-n}\}$ is unbounded. This implies that, for any $\epsilon > 0$, y^n is $< \epsilon$ for large enough n. Hence the sequence y^n converges to 0.

$\square$

A sequence of real (or rational) numbers a_n is said to be *monotone increasing*, resp. *monotone decreasing*, iff for all n, $a_n \leq a_{n+1}$, resp. $a_n \geq a_{n+1}$; it is said to be simply *monotone* if it is either monotone increasing or monotone decreasing.

Theorem A *If* $\{a_n\}$ *is a* **bounded, monotone** *sequence of* **real** *numbers, then it converges in* $\mathbb{R}$.

Proof. Suppose $\{a_n\}$ is monotone *increasing*. Since $\mathbb{R}$ is a complete ordered field, every bounded subset has a least upper bound. Since $\{a_n\}$ is bounded, we may look at its supremum M, say. We claim that $a_n \to M$. Suppose not. Then there exists an $\epsilon > 0$ such that for every $N > 0$, there exists an $n > N$ such that $|M - a_n| \geq \epsilon$. Since the sequence is monotone increasing, we then have

$$|a_n| < M - \frac{\epsilon}{2}, \quad \forall n \geq 1.$$

Then $M - \frac{\epsilon}{2}$ is an upper bound for $\{a_n\}$, and so M could not be the *least* upper bound. Contradiction! So M must be the limit of the sequence.

Now suppose $\{a_n\}$ is monotone *decreasing*. Then the sequence $\{-a_n\}$ is monotone increasing, and it is bounded because $\{a_n\}$ is. So $\{-a_n\}$ converges; then so does $\{a_n\}$.

$\square$

Here is an example. Define a sequence $\{a_n\}$ by putting

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!}.$$

It is not hard to see that this sequence is bounded. Try to give a proof. (In fact one can show that it is bounded by 3, but we do not need the best possible bound at this point.) Clearly, $a_{n+1} > a_n$, so the sequence is also monotone increasing. So we may apply the Theorem and conclude that it converges to a limit e, say, in $\mathbb{R}$. But it should be remarked that one can show with more work that e is irrational. So there is a valuable lesson to be learned here. Even though $\{a_n\}$ is a bounded, monotone sequence of rational numbers, there is no limit in $\mathbb{Q}$; one has to go to the enlarged number system $\mathbb{R}$.

A *subsequence* of a sequence $\{a_n\}$ is a subcollection indexed by a countably infinite subset J of $\mathbb{N}$. For example, $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are subsequences.

Theorem B *Suppose $\{a_n\}$ is a bounded sequence of real numbers. Then it has a subsequence which converges.*

Proof. Put
$$J = \{m \in \mathbb{N} \,|\, a_n \leq a_m \,\forall\, n \geq m\}.$$
Suppose J is an infinite set. Write its elements as $m_1, m_2, m_3, \ldots$. By definition, we have
$$a_{m_1} \geq a_{m_2} \geq am_3 \geq \ldots .$$
So the subsequence $\{a_m \,|\, m \in J\}$ (of $\{a_n\}$) is a bounded, monotone decreasing sequence, hence convergent by Theorem A. We are done in this case.

Now suppose J is finite. Pick an n_1 which is larger than any element of J, which is possible. Then we can find an $n_2 > n_1$ such that $a_{n_1} < a_{n_2}$. Again we ca find an n_3 such that $a_{n_2} < a_{n_3}$, and so on. This way we get a subsequence $\{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ which is bounded and monotone increasing, hence convergent by Theorem A.

□

2.2 Cauchy's criterion

The main problem with the definition of convergence of a sequence is that it is hard to verify it. For instance we need to have a candidate for the limit to verify (2.1.2). One wants a better way to check for convergence. As seen above, boundedness is a necessary, but not sufficient, condition, unless the sequence is also monotone (and real). Is there a necessary and sufficent condition? A nineteenth century French mathematician named Cauchy gave an affirmative answer for real sequences. His criterion also works for complex sequences.

A sequence $\{a_n\}$ is said to be a *Cauchy sequence* iff we can find, every positive ϵ, an $N > 0$ such that

$$|a_m - a_n| < \epsilon \quad \text{whenever} \quad n, m > N. \tag{2.2.1}$$

This is nice because it does not mention any limit.

Lemma 2.2.2 *Every convergent sequence is Cauchy.*

Proof. Suppose $a_n \to L$. Pick any $\epsilon > 0$. Then by definition, we can find an $N > 0$ such that for all $n > N$, we have
$$|L - a_n| < \frac{\epsilon}{2}$$
Then for $n, m > N$, the triangle inequality gives
$$|a_m - a_n| \leq |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Lemma 2.2.3 *A complex sequence $\{z_n = x_n + iy_n\}$, with x_n, y_n real, is Cauchy iff $\{x_n\}$ and $\{y_n\}$ are convergent.*

Proof. Suppose $\{x_n\}, \{y_n\}$ are both Cauchy. Pick any $\epsilon > 0$. Then we can find an $N > 0$ such that for all $m, n > N$,

$$|x_m - x_n| < \frac{\epsilon}{2}, \quad \text{and} \quad |y_m - y_n| < \frac{\epsilon}{2}.$$

Since $z_m - z_n$ is $(x_m - x_n) + i(y_m - y_n)$ and $|i| = 1$, , we get by the triangle inequality,

$$|z_m - z_n| \leq |x_m - x_n| + |y_m - y_n| < \epsilon.$$

Conversely, suppose that $\{z_n\}$ is Cauchy. Recall that for any complex number $z = x + iy$, $|x|$ and $|y|$ are both bounded by $\leq \sqrt{x^2 + y^2}$, i.e., by $|z|$. This immediately implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy.

□

Here comes the beautiful result of Cauchy:

Theorem C *Every Cauchy sequence of real (resp. complex) numbers converges in* $\mathbb{R}$ *(resp.* $\mathbb{C}$*).*

Proof. By Corollary 2.1.3, a sequence $\{z_n = x_n + iy_n\}$ of complex numbers is convergent iff the real sequences $\{x_n\}$ and $\{y_n\}$ are convergent. By Lemma 2.2.3, $\{z_n\}$ is Cauchy iff $\{x_n\}, \{y_n\}$ are. So it suffices to prove Theorem C for *real* Cauchy sequences $\{a_n\}$.

First note that such a sequence must be *bounded*. Indeed, fix an $\epsilon > 0$ and find an $N > 0$ for which (2.2.1) holds. Then for all $n > N$, a_n lies between $a_{N+1} - \epsilon$ and $a_{N+1} + \epsilon$. Then the set $\{a_n \mid n > N\}$ is bounded. Then so is the entire sequence because $\{a_1, \ldots, a_N\}$ is finite.

Now since $\{a_n\}$is a bounded, real sequence, we can apply Theorem B and find a subsequence $\{a_{m_1}, a_{m_2}, \ldots\}$ which converges, say to the limit L. We claim that the original sequence $\{a_n\}$ also converges to L. Indeed, let ϵ be an arbitrary positive number. Then we can find an $N > 0$ such that for all $m_k, n > N$,

$$|L - a_{m_k}| < \frac{\epsilon}{2} \quad \text{and} \quad |a_{m_k} - a_n| < \frac{\epsilon}{2}.$$

Applying the triangle inequality we get

$$|L - a_n| < \epsilon \quad \forall\, n > N,$$

proving the claim.

□

2.3 Construction of Real Numbers revisited

Given any *Cauchy sequence* $\{a_n\}$ **of rational numbers**, we may of course view it as a real sequence and apply Theorem C to deduce that it converges to a limit L, say. But this L need not be rational. This leads us to wonder if *all* the real numbers could be constructed by adjoining limits of Cauchy sequences of rational numbers. This is essentially correct, and this approach is quite different from that given in chapter 1. Now let us explain how it is done.

Let us begin by pretending that we do not know what real numbers are, but that we know everything about rational numbers.

The first thing to note is that different Cauchy sequences of rational numbers might have the same limit. For example, the sequences $\{0, 0, 0, \dots\}$ and $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ have 0 as their limit. So we define an **equivalence** between any pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \in \mathbb{Q}$ are as follows:

$$\{a_n\} \sim \{b_n\} \Leftrightarrow \lim_{n\to\infty} (a_n - b_n) = 0. \tag{2.3.1}$$

It will be left as a good exercise for the students to check that $\sim$ is indeed an equivalence relation in the sense of chapter 0.

Denote the equivalence class of any $\{a_n\}$ by $[a_n]$. It is not hard to check that given two classes $[a_n]$ and $[b_n]$, they are either the same or disjoint, i.e., no rational Cauchy sequence $\{c_n\}$ can belong to two distinct classes.

Define a **real number** to be an equivalence class $[a_n]$ as above. Denote by $\mathbb{R}$ the collection of all real numbers in this sense. We then have a natural, mapping

$$\mathbb{Q} \to \mathbb{R}, \; q \to [q, q, q, \dots],$$

where $[q, q, q, \dots]$ is the class of the **stationary sequence** $\{q, q, q, \dots\}$ (which is evidently Cauchy). This mapping is one-to-one because if q, q' are unequal rational numbers, then $\{q, q, q, \dots\}$ cannot be equivalent to $\{q', q', q', \dots\}$.

Consequently, we have 0 and 1 in $\mathbb{R}$.

We know that the sum and product of Cauchy sequences are again Cauchy. So we may set

$$[a_n] \pm [b_n] = [a_n \pm b_n], \quad \text{and} \quad [a_n] \cdot [b_n] = [a_n b_n].$$

It is immediate that one has commutativity, associativity and distributivity.

It remains to define a multiplicative inverse of a non-zero element ρ of $\mathbb{R}$. Let $\{a_n\}$ be a Cauchy sequence belonging to ρ. Since ρ is non-zero, all but a finite number of the a_n must be non-zero. Let $\{a_{m_n}\}$ be the subsequence consisting of all the non-zero terms of $\{a_n\}$. Then this subsequence also represents ρ, and we may set

$$\rho^{-1} = [a_{m_n}^{-1}].$$

Check that this gives a well defined multiplicative inverse.

Now it is more or less obvious that $\mathbb{R}$ is a field under these operations, containing $\mathbb{Q}$ as a subfield.

Next comes the notion of *positivity*. Define a real number ρ to be positive iff for some Cauchy sequence $\{a_n\}$ representing ρ, we can find an $N > 0$ such that a_n is positive for all $n > N$. Check that this intuitive notion is the right one.

Armed with this powerful notion, one sees that $\mathbb{R}$ becomes an **ordered field**. By definition, the ordering on $\mathbb{R}$ is compatible with the usual one on $\mathbb{Q}$.

Finally, every Cauchy sequence converges in $\mathbb{R}$. This is a nice consequence of our construction. When ρ is represented by a Cauchy sequence $\{a_n\}$, the sequence is forced to converge to ρ.

Now we will end this section by discussing how one can mathematically define real numbers via their **decimal expansions**, which everyone intuitively understands. Here is what one does.

Define a **real number** to be a pair $(m, \{a_n\})$, where m is an integer and $\{a_n\}$ is a sequence of non-negative integers, with the understanding that a_n is not all 9 after some index N. One often writes

$$(m, \{a_n\}) \;=\; m + \sum_{n=1}^{\infty} a_n 10^{-n}.$$

If we have a rational number of the form $m + \sum_{n=1}^{r} a_n$, we will identify it with $(m, \{a_n\})$ with $a_n = 0$ for all $n > r$.

Define an ordering by setting

$$(m, \{a_n\}) \;<\; (k, \{b_n\}) \;\Leftrightarrow\; m < k \quad \text{or} \quad m = k, \exists N \,\text{s.t.}\, a_N < b_N,\, a_n = b_n \,\forall\, n < N.$$

This leads quickly to the **least upper bound property**, so we can take the suprema of bounded sets.

Given any real number $\rho = (m, \{a_n\})$, we will define, for each $r \geq 1$, its **truncation** at stage r to be

$$\rho_r \;=\; m + \sum_{n=1}^{r} a_n \;\in\; \mathbb{Q}.$$

Evidently,

$$\rho \;=\; \sup\{\rho_r \,|\, r \geq 1\}.$$

So it is natural to set, for real numbers ρ, ρ',

$$\rho \pm \rho' \;=\; \sup\{\rho_r \pm \rho'_r\}$$

and

$$\rho \cdot \rho' \;=\; \sup\{\rho_k \cdot \rho'_k\}.$$

The definition of the multiplicative inverse of a non-zero ρ is left as an exercise.

2.4 Infinite series

Given a sequence $\{a_n\}$, we will often want to know if we can sum the terms of the sequence, denoted

$$S \;=\; \sum_{n=1}^{\infty} a_n, \tag{2.4.1}$$

and get something meaningful. Such an infinite sum is called an *infinite series*.

A natural way to proceed is as follows. We know how to sum, or any $n \geq 1$, the first n terms of the sequence, namely

$$s_n = \sum_{m=1}^{n} a_n. \tag{2.4.2}$$

This way we get another sequence $\{s_n\}$, called the sequence of *partial sums*. (One calls s_n the nth partial sum.) We will say that the original sequence $\{a_n\}$ is *summable*, or that the infinite series (2.4.1) is convergent, iff the sequence of partial sums converges.

Let us first look at the example of a **geometric series**. Fix any real number x with $|x| < 1$, and put

$$a_n = x^{n-1} \, (\forall \, n \geq 1),$$

and

$$S = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \ldots \tag{2.4.3}$$

We claim that

$$s_n = \frac{1 - x^n}{1 - x}.$$

Indeed, we have

$$(1-x)s_n = (1-x)(1+x+x^2+\ldots+x^{n-1}) = (1-x)+(x-x^2)+\ldots+(x^{n-1}-x^n) = 1-x^n,$$

proving the claim.

Now by Lemma 2.1.5 (and the discussion preceding it), we have

$$x^n \to 0 \quad \text{since} \quad |x| < 1.$$

Consequently,

$$S = \lim_{n\to\infty} s_n = \frac{1}{1-x}. \tag{2.4.2}$$

Check that the series is divergent if $|x| \geq 1$.

Lemma 2.4.3 *If $\{a_n\}, \{b_n\}$ are summable sequences, then so is $\{c_1a_n + c_2b_n\}$ for any pair of scalars c_1, c_2.*

The proof is straight-forward and will be left as an exercise. The reader should take note, however, that nothing is claimed here about the product of two sequences.

We will say that a series $\sum_{n\geq 1} a_n$ is *absolutely convergent* if the series $\sum_{n\geq 1} |a_n|$ is convergent. In this case one also says that the sequence $\{a_n\}$ is *absolutely summable*. This notion is important and makes us want to understand well the summability of positive sequences, which will be our focus in the next section. It should be noted however (see section 2.6) that there are summable sequences, which are not absolutely summable.

One of the most important reasons for studying absolutely summable sequences is that it allows one to study the product of two sequences. This is encapsulated in the following Proposition, which we will quote without proof.

Proposition 2.4.4 *Suppose $\{a_n\}$, $\{b_n\}$ are two absolutely summable sequences. Consider the sequence $\{c_n\}$ with*

$$c_n = \prod_{i+j=n} a_i b_j.$$

Then $\{c_n\}$ is absolutely summable and moreover,

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n.$$

2.5 Tests for Convergence

We will concentrate mainly on the summability of sequences with *non-negative* terms.

The first thing one notices about such a sequence $\{a_n\}$ with each a_n non-negative, is that

$$s_1 \leq s_2 \leq \ldots \leq s_n \leq \ldots,$$

i.e., $\{s_n\}$ is monotone increasing. In view of Theorem A, we then get the following

Lemma 2.5.1 *A sequence of non-negative numbers is summable iff the associated sequence $\{s_n\}$ of partial sums is bounded.*

The first test for convergence is given by the following

Proposition 2.5.2 (Comparison test) *Let $\{a_n\}$, $\{b_n\}$ be non-negative sequences of real numbers, such that*

(i) $a_n \leq b_n$ for all $n \geq 1$; and

(ii) The sequence $\{b_n\}$ is summable.

Then $\{a_n\}$ is summable.

Proof. Let s_n, t_n denote the n-th partial sums of $\{a_n\}$, $\{b_n\}$ respectively. Then we have (for all n

$$s_n \leq t_n.$$

Since by hypothesis the sequence $\{b_n\}$ is summable, the sequence $\{t_n\}$ converges. Then the t_n, and hence the s_n, must be bounded. So the sequence $\{a_n\}$ is summable by Lemma 2.5.2. □

Here comes the second test.

Proposition 2.5.4 (Ratio test)) *Let $\{a_n\}$ be a sequence of non-negative terms. Suppose the sequence $\{a_{n+1}/a_n\}$ of ratios converges to a real number r.*

(i) If $r < 1$, then $\{a_n\}$ is summable;

(ii) If $r > 1$, then $\{a_n\}$ is not summable, i.e., the series $\sum\limits_{n\geq 1} a_n$ diverges.

It is important to note that this test says nothing when $r = 1$.

Proof. Suppose r is less than 1. Pick any $\epsilon > 0$ with $r + \epsilon < 1$. Write $t = r + \epsilon$, so that $r < t < 1$. Then as $\{a_{n+1}/a_n\}$ converges to r, we can find an $N > 0$ such that for all $n > N$, $|r - a_{n+1}/a_n| < \epsilon$. This implies that

$$n > N \Rightarrow \frac{a_{n+1}}{a_n} < t.$$

In other words,

$$a_{N+2} < a_{N+1}t,\, a_{N+3} < a_{N+2}t < a_{N+1}t^2, \ldots$$

We can rewrite it as saying that for all $j \geq 1$, we have

$$a_{N+j} < a_{N+1}t^{j-1},$$

so that

$$\sum_{j=1}^{\infty} a_{N+j} < a_{N+1} \sum_{j\geq 1}^{\infty} t^{j-1}.$$

This converges by the comparison test, because the geometric series $\sum_{j\geq 1} t^{j-1}$ occurring on the right is convergent as t is positive and < 1 (with limit $\frac{1}{1-t}$). Consequently, the (partial) series $\sum\limits_{n>N} a_n$, and hence the whole series $\sum\limits_{n\geq 1} a_n$, converges.

When $r > 1$, one sees easily that the sequence of partial sums is not bounded, hence divergent.

□

Given a positive real number x and a positive integer n, there is a unique positive real number, denoted $x^{1/n}$, which is an n-th root of x, i.e., $(x^{1/n})^n = x$. We will also set $0^{1/n} = 0$.

Proposition 2.5.5 (Root test) *Let $\{a_n\}$ be a sequence of non-negative real numbers. Suppose the associated root sequence $\{a_n^{1/n}\}$ is convergent with limit R, Then the following hold:*

(i) If $R < 1$, then the series $\sum\limits_{n\geq 1} a_n$ converges;

(ii) If $R > 1$, then the series $\sum\limits_{n\geq 1} a_n$ diverges.

Just as in the Ratio test, this test says nothing when $R = 1$.

Proof. Suppose $R < 1$. Pick any $\epsilon > 0$ with $R + \epsilon < 1$. Write $t = R + \epsilon$, so that $R < t < 1$. Then as $\{a_n^{1/n}\}$ converges to R, we can find an $N > 0$ such that for all $n > N$, $|R - a_n^{1/n}| < \epsilon$. This implies that

$$n > N \Rightarrow a_n < t^n.$$

Hence

$$\sum_{n>N} a_n < \sum_{n>N} t^n < \sum_{n\geq 1} t^n.$$

The (partial) series on the left now converges by the comparison test, because the geometric series $\sum_{n\geq 1} t^n$ occurring on the right is convergent (as t is positive and < 1). Consequently, the complete series $\sum\limits_{n\geq 1} a_n$ also converges.

When $R > 1$, the sequence of partial sums is not bounded, hence divergent.

□

2.6 Alternating series

One of the most beautiful formulas in Mathematics is the formula of Leibniz for $\pi/4$ given at the beginning of this chapter. It is one of the glorious examples of what are known as *alternating series*, i.e., whose terms alternate in sign, which converge but not absolutely. Some people use the terminology *conditional convergence* for this phenomenon. Leibniz was amazing in that he not only found a slew of such formulas, he also discovered a theorem which gives precise conditions on an alternating series to be convergent. Let us now state and prove it.

Theorem D (*Leibniz*) *Let $\{b_n\}$ be a sequence of* **non-negative**, **monotone decreasing** *sequence such that*

$$\lim_{n\to\infty} b_n = 0.$$

Then the alternating series

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \ldots$$

is convergent.

Proof. Put $a_n = (-1)^{n+1} b_n$ and

$$s_n = a_1 + a_2 + \ldots + a_n = b_1 - b_2 + \ldots + (-1)^{n+1} b_n.$$

We have to show that the sequence $\{s_n\}$ of partial sums converges. First note that for any $n \geq 1$,

$$s_{2n} = s_{2n-1} - b_{2n} \leq s_{2n-1}, \tag{2.6.1}$$

since the b_j are non-negative. We also have (for all $n \geq 2$)

$$s_{2n} \;=\; s_{2n-2} + b_{2n-1} - b_{2n} \;\leq\; s_{2n-2}, \tag{2.6.2}$$

and

$$s_{2n-1} \;=\; s_{2n-3} - b_{2n-2} + b_{2n-1} \;\geq\; s_{2n-3}, \tag{2.6.3}$$

both since the b_j are monotone decreasing.

Now let i, j be natural numbers such that i is even and j odd. Then we may pick an $n > 0$ such that $2n \geq i$ and $2n - 1 \geq j$. Applying (2.6.1), (2.6.2) and (2.6.3), we deduce that

$$s_i \;\leq\; s_{2n} \;\leq\; s_{2n-1} \;\leq\; s_j. \tag{2.6.4}$$

In particular, s_i is bounded from above, for any even index i, by $s_1 = b_1$. Similarly, for any odd index j, s_j is bounded from below by $s_2 = b_1 - b_2$.

The inequalities given by (2.6.2) and (2.6.3) show that the subsequence $\{s_{2n}\}$ is monotone increasing and $\{s_{2n-1}\}$ is monotone decreasing. Since the former is bounded from above, and the latter from below, they both converge, say to respective limits ℓ and L, i.e., we have

$$\lim_{n\to\infty} s_{2n} \;=\; \ell \;\leq\; L \;=\; \lim_{n\to\infty} s_{2n-1}.$$

The assertion $\ell \leq L$ follows from (2.6.4).

Finally, note that by definition,

$$s_{2n} - s_{2n-1} \;=\; -b_{2n}.$$

Since the b_n converges to zero as $n \to \infty$, we see that the subsequences $\{s_{2n}\}$ and s_{2n-1} must both converge to the same limit, i.e., $\ell = L$. Consequently,

$$\lim_{n\to\infty} s_n \;=\; L.$$

$\square$

Here are some examples of alternating series:

$$S_1 \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \tag{2.6.5}$$

$$S_2 \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \tag{2.6.6}$$

$$S_3 \;=\; \pi - \frac{\pi^2}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots \tag{2.6.7}$$

Clearly, $\{\frac{1}{n}\}$ and $\{\frac{1}{2n-1}\}$ are non-negative, decreasing sequences converging to zero. So we may apply Leibniz's theorem above and conclude that S_1 and S_2 converge. Even though we will not evaluate these limits here, let us note that

$$S_1 = \log 2 \quad \text{and} \quad S_2 = \frac{\pi}{4}. \tag{2.6.8}$$

Let us now turn to (2.6.7). We need to know that

$$\frac{\pi^n}{n!} \leq \frac{\pi^{n-1}}{(n-1)!} \tag{2.6.9}$$

and that

$$\frac{\pi^n}{n!} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.6.10}$$

Since π is known to be < 4, (2.6.9) is satisfied for all $n \geq 4$, and when we study exponential functions later in the course, we will see the truth of (2.6.10). So by Leibniz's theorem, the series starting with $n = 4$ converges. So S_3 itself converges. The amazing things is that one can show (to come later)

$$S_3 = 0.$$