Notes on Calculus

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11 Uniform convergence, Taylor series, Complex series

In this chapter we will lay down the basic results underlying the convergence – **pointwise** and **uniform** of infinite series of functions. No proofs will be given, but the student should feel free to make use of them, of course taking care to apply them only when the relevant hypotheses are satisfied. We will use this theory to study the ever important **Taylor series expansions** of various functions, including the exponential, trigonometric and simple rational functions, as well as the logarithm. We will at the end extend these functions to the domain of complex numbers to the extent possible.

11.1 Infinite series of functions, convergence

Let $\{f_n\}$ be a sequence of functions on a subset X of $\mathbb{R}$ or $\mathbb{C}$. We will say that this sequence is **pointwise convergent** with limit f on X iff for every $x \in X$, the sequence $\{f_n(x)\}$ of numbers converges to $f(x)$. In other words, for every $\varepsilon > 0$, there is a positive number $N(x)$ such that

(11.1.1) $$n > N(x) \implies |f(x) - f_n(x)| < \varepsilon.$$

It is natural to wonder if $N(x)$ can be taken to be a number N, say, which is **independent of** x. In such a case, we will say that f_n is **uniformly convergent** with limit f on X.

Uniform convergence is a very important concept for Calculus. To elaborate, we have the following useful result:

Theorem 11.1.2 *Let $\{f_n\}$ be a sequence of functions on a finite interval I in $\mathbb{R}$, and let f be a function on the same interval I. Then*

*(a) (**Integration**) If $f_n \to f$ uniformly on I, and if f_n, f are integrable on I, then*

$$\int_I f(x)dx = \lim_{n\to\infty} \int_I f_n(x)dx.$$

*(b) (**Continuity**) If $f_n \to f$ uniformly on I, and if the f_n are continuous, then f is also continuous.*

*(c) (**Differentiability**) Suppose $\{f_n\}$ converge pointwise to f, where each f_n is differentiable with f_n' integrable. Further assume that the sequence $\{f_n'\}$ converges uniformly to a continuous function ϕ on I. Then f is differentiable and*

$$f'(x) = \lim_{n\to\infty} f_n'(x),$$

for all $x \in I$.

Now consider any **infinite sum of functions** on a subset X of $\mathbb{R}$ or $\mathbb{C}$, given as

(11.1.3) $$\sum_{n=0}^{\infty} f_n.$$

To know if this makes sense, we need to look at the **partial sums** s_n defined by

$$s_n = \sum_{j=0}^{n} f_j. \tag{11.1.4}$$

We will say that the sum S converges uniformly, resp. **pointwise**, to a function S on X iff the sequence $\{s_n\}$ converges to S uniformly, resp. pointwise, on X.

Theorem 11.1.2 has a natural analog for infinite sums.

Theorem 11.1.5 *Suppose we are given* $\sum_{n=0}^{\infty} f_n, S$ *on an interval* I *in* $\mathbb{R}$*.*

*(a) (***Integrability***) If* $\sum_{n=0}^{\infty}$ *converges uniformly to* S *in* I*, and if the* f_n, f *are integrable there, then*

$$\int_I S(x)dx = \sum_{n=1}^{\infty} \int_I f_n(x)dx.$$

*(b) (***Continuity***) If* $\sum_{n=0}^{\infty}$ *converges uniformly to* S *in* I*, and if the* f_n *are continuous, the so is* S*.*

*(c) (***Differentiability***) Suppose* $\sum_{n=1}^{\infty} f_n$ *converges pointwise to* S*, where each* f_n *is differentiable with* f_n' *integrable. Further assume that the sum* $\sum_{n=1}^{\infty} f_n'$ *converges uniformly to a continuous function* T *on* I*. Then* S *is differentiable and*

$$S'(x) = \sum_{n=1}^{\infty} f_n'(x),$$

for all $x \in I$*.*

Here is a very helpful test to determine if a given infinite series converges uniformly or not.

Theorem 11.1.6 (Weierstrass's test) *Let* $\{f_n\}$ *be a sequence of functions on an interval* I *in* $\mathbb{R}$*, and let* $\{M_n\}$ *be a sequence of positive numbers such that*

(i) $|f_n(x)| \leq M_n$ *for all* n*, and for all* x*; and*

(ii) the infinite series of numbers $\sum_{n=1}^{\infty} M_n$ *converges.*

Then

(a) For all $x \in I$*, the series* $\sum_{n=1}^{\infty} f_n(x)$ *converges absolutely; and*

(b) the sum $\sum_{n=1}^{\infty} f_n$ *converges uniformly on* I *to the function*

$$S(x) = \sum_{n=1}^{\infty} f_n(x).$$

Here is an **example** illustrating the usefulness of this test. For any integer $k \geq 2$, consider the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^k}. \tag{11.1.7}$$

Put

$$M_n = \frac{1}{n^k}.$$

Then, since $\sin nx$ is always bounded between -1 and 1,

$$\left| \frac{\sin nx}{n^k} \right| \leq M_n \quad \forall\, n \geq 1.$$

Moreover, since $k > 1$,

$$\sum_{n=1}^{\infty} M_n$$

converges. So we may apply Weierstrass's test and conclude that the series (11.1.7) converges.

Note that if we take the above series, but with $k = 1$, then the test does not apply as the sum of $1/n$ diverges. But that does not mean that the series $\sum_{n\geq 1} \sin nx\, n^{-1}$ is divergent for all x. It only means that the method does not apply. In fact one can show, with some trouble, that this series is convergent for x in $[t, \pi - t]$, for any $t > 0$. Of course it is problematic when x is a multiple of π.

11.2 Taylor series

A series of the form

$$\sum_{n=0}^{\infty} a_n x^n \tag{11.2.1}$$

is called a **power series**.

Since the power functions $x \to x^n$ are integrable and differentiable, in fact any number of times, we may specialize Theorem 11.1.5 to the case of power series and obtain the following:

Theorem 11.2.2 For any sequence $\{a_n\}$ of real numbers, put

$$S(x) = \sum_{n=0}^{\infty} a_n x^n$$

(i) If the series is **uniformly** convergent in a closed interval $[a, b]$, then S is integrable on $[a, b]$ and

$$\int_a^b S(x)dx = \lim_{n\to\infty} a_n \frac{b^{n+1} - a^{n+1}}{n+1}.$$

(ii) Suppose the series converges pointwise, and more importantly, its series of derivatives, namely

$$T(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

converges **uniformly** in $[a, b]$. Then S is differentiable with derivative T.

A particular kind of power series, which is of great utility, is the *so called* **Taylor series** (at 0), which is of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{11.2.3}$$

for some infinitely differentiable function f.

Theorem 11.2.2 says that if a power series is uniformly convergent we can integrate it term by term to get its integral, and if the series of derivatives is also uniformly convergent, then we can differentiate it term by term to get the derivative. So, roughly speaking, when we have uniform convergence of the appropriate power series we are allowed to integrate and differentiate term by term. What is left to do is to find a criterion to know when a power series is uniformly convergent. We also want this criterion to tell us when the series and the formally derived series both converge uniformly. Here is the completely satisfactory result giving us what we need:

Theorem 11.2.4 *Let $\{a_n\}$ be a sequence of real numbers. Suppose there is a positive real number c such that*

$$\sum_{n=0}^{\infty} a_n c^n$$

converges. Pick any positive number $b < c$. Then each of the power series

$$S(x) = \sum_{n=0}^{\infty} a_n x^n$$

and

$$T(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

converges absolutely and uniformly in $[-b, b]$. Moreover,

$$S'(x) = T(x) \quad \forall\, x \in (-c, c).$$

The proof of this result, which we will skip, makes clever use of the Weierstrass test and Theorem 11.2.2.

Let us try to understand the **exponential function** using this. We have seen that

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n(e^x;0),$$

with

$$|R_n(e^x;0)| \;\leq\; e^x \frac{|x|^{n+1}}{(n+1)!}.$$

Consequently, for any positive c, the remainder term $R_n(e^c;0)$ goes to zero as n goes to infinity. This gives us the convergent infinite series expression

$$e^c \;=\; 1 + c + \frac{c^2}{2!} + \frac{c^3}{3!} + \ldots$$

Applying Theorem 11.2.4 we see then that the **Taylor series**

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{11.2.5}$$

converges absolutely and uniformly on $[-c, c]$ for any $c > 0$. Moreover, its derivative is given by differentiating the series term by term, which gives back e^x, as expected.

By a similar argument, we also get the Taylor series expressions for the sine and cosine functions, valid for all x:

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \tag{11.2.5 $-$ i}$$

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \tag{11.2.5 $-$ ii}$$

We can also apply Theorem 11.2.4 and differentiate the expression for $\sin x$ term by term, and as expected, one gets the series expression for $\cos x$.

The series

$$\arctan x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \tag{11.2.7}$$

is, as we saw before, convergent at $x = 1$. Thus by Theorem 11.2.4, it converges absolutely in $-1 < x < 1$ and uniformly in $[-b, b]$ for any positive $b < 1$.

We claim that this Taylor series for $\arctan x$ does not converge at any $x > 1$. Indeed, if it did, then by Theorem 11.2.4, it would converge absolutely at $x = 1$ which it does not. So the number 1, which some call the *radius of convergence* of the Taylor series (11.2.7), is the boundary beyond which there is no convergence.

Taking the derivative of the series (11.2.7) term by term, and applying Theorem 11.2..4, we get (as expected)

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots, \tag{11.2.8}$$

which is valid in $\{|x| < 1\}$. This certainly does not converge at $x = -1$. Even at $x = 1$ the series makes no sense, but the left hand side makes sense there and equals $\frac{1}{2}$.

Incidentally, the Taylor series for $\arctan x$ converges at $x = -1$, with

$$\arctan(-1) = -\arctan 1 = -\frac{\pi}{4}.$$

It should be noted, despite what the arctan function may suggest, that in Theorem 11.2.4, the absolute convergence of $S(x)$ is asserted only for $|x| < c$, and one can say nothing in general about the convergence, or the lack of it, at $x = -c$. Sometimes it does happen that $S(x)$ is divergent at $-c$. The simplest example illustrating this is the Taylor series for $\log(1+x)$ at $x = 0$, which we claim to be

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \tag{11.2.9}$$

absolutely convergent in $-1 < x < 1$ and uniformly convergent in $[-b, b]$ for any positive $b < 1$. One way to see this is the following:

Since $\log(1+x)$ is (defined and) differentiable in $x > -1$ with derivative $\frac{1}{1+x}$, we could find the Taylor series expansion of $\frac{1}{1+x}$ first and then differentiate.

The identity (11.2.8) implies that

$$\frac{1}{1+x} = 1 - x + x^3 - x^4 + \dots \tag{11.2.10}$$

which is absolutely convergent in $0 \le x < 1$. Then it is also absolutely convergent in $(-1, 1)$. Pick any positive $b < 1$. Then since the series (11.2.10) converges at c for any c with $b < c < 1$. So we may apply Theorem 11.2.4 and conclude that (11.2.10) converges uniformly in $[-b, b]$. We can also differentiate term by term, by this Theorem, and obtain the desired expansion (11.2.9), which is valid in $-1 \le x \le 1$.

At $x = 1$, the series (11.2.9) is alternating and converges to $\log 2$. But it is important to note, however, that it is divergent at $x = -1$.

A consequence of (11.2.9) is the series for $\log(1-x)$, given by

$$-\log(1-x) = \log\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \tag{11.2.11}$$

A very useful thing to remember, when computing the Taylor series of various functions is the following

Lemma 11.2.12 *For $j = 1, 2$, consider power series*

$$S_j(x) \;=\; \sum_{n=0}^{\infty} a_{n,j} x^n$$

If they are both uniformly convergent in some $[a, b]$, then for all scalars α, β, the power series

$$\sum_{n=0}^{\infty} (\alpha a_{n,1} + \beta a_{n,2}) x^n$$

converges uniformly in $[a, b]$ to $\alpha S_1(x) + \beta S_2(x)$.

In particular, the sum or difference of two uniformly convergent power series is again uniformly convergent.

11.3 Complex power series

It is very important to understand the power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{11.3.1}$$

where the coefficients a_n are complex numbers and z is in $\mathbb{C}$.

We will say that such a series is absolutely convergent if the associated (non-negative) real series

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n \tag{11.3.2}$$

converges. For example, the **complex exponential function**

$$e^z \;=\; 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \tag{11.3.3}$$

is absolutely convergent for all z because the corresponding series of absolute values, namely

$$1 + |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \ldots$$

is convergent, it being the Taylor series of the real exponential $e^{|z|}$.

Here is the main result of this section.

Theorem 11.3.4 *Suppose the complex power series*

$$S(z) \;=\; \sum_{n=0}^{\infty} a_n z^n$$

converges at some non-zero complex number z_0. Then either this series is absolutely convergent for all z in $\mathbb{C}$, or there is a positive real number $R > 0$ such that

$S(z)$ is absolutely convergent in $\{|z| < R\}$; and

$S(z)$ diverges for any z with $|z| > R$.

Moreover, $S(z)$ is differentiable at any z with $|z| < R$ with derivative

$$S'(z) \;=\; \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

R is called the **radius of convergence** of the power series. Note that the Theorem says nothing about the convergence at the points z with $|z| = 1$. Some people would say that the radius of convergence is infinite when $S(z)$ is absolutely convergent everywhere in $\mathbb{C}$.

One can define the complex extensions of the various functions we have encountered like the sine and cosine functions in terms of their power series. To be specific we have

$$\sin z \;=\; z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \tag{11.3.5}$$

$$\cos z \;=\; 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \tag{11.3.6}$$

Both these power series have infinite radii of convergence.

It is crucial to note, by comparing the power series expansions, that

$$e^{iz} \;=\; \cos z + i \sin z. \tag{11.3.7}$$

This leads to the identities

$$\sin z \;=\; \frac{e^{iz} - e^{-iz}}{2i} \tag{11.3.8}$$

and

$$\cos z \;=\; \frac{e^{iz} + e^{-iz}}{2}. \tag{11.3.9}$$

We will be remiss if we forget the **complex logarithm**

$$\log(1+z) \;=\; z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \tag{11.3.10}$$

which is absolutely convergent in $\{|z| < 1\}$. We have already seen that it converges at $z = 1$ and diverges at $z = -1$.

What we now have a is a very interesting situation for $\log(1+z)$. On the one hand, it makes sense at any complex number z with $|z| < 1$. On the other, it makes sense whenever

$1+z$ is a positive real number, though it is not given by its Taylor series (11.3.10) if $|z| > 1$. Thus $\log(1+z)$ is defined on the following strange-shaped subset of $\mathbb{C}$:

$$\{z \in \mathbb{C} \mid |z| < 1 \quad \text{or} \quad z \in (-1, \infty) \subset \mathbb{R}\}. \tag{11.3.11}$$

The natural question which arises immediately is whether this function can be given sense at all points of $\mathbb{C}$, or at least on a larger subset. The quick answer is that it can be defined on the *complement* in $\mathbb{C}$ of the half-line $(-\infty, -1] \subset \mathbb{R}$. On all of $\mathbb{C}$ it can only be defined as a multi-valued function. To understand all these intriguing matters better, one will need to understand a fundamental mathematical concept called *analytic continuation*, and for that one will need to take a course in *Complex Variables*.