Notes on Calculus

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10 Taylor's theorem, Polynomial approximations

Polynomials are the nicest possible functions. They are easy to differentiate and integrate, which is also true of the basic trigonometric functions, but more importantly, polynomials can be evaluated at any point, which is not true for general functions. So what one does in practice is to approximate any function f of interest by polynomials. When the approximation is done by *linear polynomials*, then it is called a *linear approximation*, which pictorially corresponds to *linearizing* the graph of f. It turns out that the more times one can differentiate f, the higher is the degree of the polynomial one can approximate it with, and more importantly, the better the approximation becomes, as one sees it intuitively. There is only one main theorem here, due to Taylor, but it is omnipresent in all the mathematical sciences, with a number of ramifications, and should be understood precisely.

10.1 Taylor polynomials

Suppose f is an N-times differentiable function on an open interval I. Fix any point a in I. Then for any non-negative integer $n \leq N$, the **nth Taylor polynomial of f at $x = a$** is given by

$$(10.1.1) \qquad p_n(f(x); a) \;=\; \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j,$$

where $f^{(j)}(a)$ denotes the jth derivative of f at a. By convention, $f^{(0)}(a)$ just denotes $f(a)$. (f is the 0th derivative of itself!)

The coefficients $\frac{f^{(j)}(a)}{j!}$ are called the **Taylor coefficients of f at a.**

The definition has been rigged so that the following holds:

Lemma 10.1.2 *Suppose f is itself a polynomial, i.e.,*

$$f(x) \;=\; a_0 + a_1 x + \ldots + a_m x^m,$$

for some integer $m \geq 0$. Then f is infinitely differentiable (which means it can be differentiated any number of times), and

$$p_n(f(x); 0) \;=\; \begin{cases} a_0 + a_1 x + \ldots + a_n x^n, & \text{if} \quad n < m \\ a_0 + a_1 x + \ldots + a_m x^m, & \text{if} \quad n \geq m \end{cases}$$

Proof. Clearly, f is differentiable any number of times and moreover, $f^{(n)}(x)$ vanishes if $n > m$. So we have only to show that for $n \leq m$,

$$(10.1.3) \qquad f^{(n)}(0) \;=\; n! a_n.$$

When $m = 0$ this is clear. So let $m > 0$ and assume by induction that (10.1.3) holds for all polynomials of degree $m-1$ and $n \le m-1$. Define a polynomial $g(x)$ by the formula

$$f(x) \;=\; a_0 + xg(x).$$

Then

$$g(x) \;=\; \sum_{j=0}^{m-1} a_{j+1}x^j$$

and by the inductive hypothesis,

$$g^{(n)}(0) \;=\; n!a_{n+1} \tag{10.1.4}$$

for all non-negative $n \le m-1$. But by the *product rule*,

$$f'(x) \;=\; g(x) + xg'(x),\; f''(x) \;=\; 2g'(x) + xg''(x), \ldots$$

By induction, we get

$$f^{(n)}(x) \;=\; ng^{(n-1)}(x) + xg^{(n)}(x),$$

so that

$$f^{(n)}(0) \;=\; ng^{(n-1)}(0)\; \forall\, n \le m, n \ge 1. \tag{10.1.5}$$

The identity (10.1.3), and hence the Lemma, now follow by combining (10.1.4) and (10.1.5). □

Lemma 10.1.6 (**Linearity**) *Let f, g be n-times differentiable at a, and let α, β be arbitrary scalars. Then*

$$p_n(\alpha f(x) + \beta g(x); a) \;=\; \alpha p_n(f(x); a) + \beta p_n(g(x); a).$$

This is easy to prove because the derivative is linear. In particular, we have

$$\frac{(\alpha f + \beta g)^{(j)}(a)}{j!} \;=\; \alpha \frac{f^{(j)}(a)}{j!} + \beta \frac{g^{(j)}(a)}{j!}.$$

It is helpful to look at some **examples**:

(**1**): Let

$$f(x) \;=\; \sin x,$$

which is infinitely differentiable, with

$$f'(x) \;=\; \cos x,\; f''(x) \;=\; -\sin x \;=\; -f(x).$$

Thus

$$f^{(n)}(x) \;=\; \begin{cases} (-1)^k \sin x, & \text{if} \quad n = 2k \\ (-1)^k \cos x, & \text{if} \quad n = 2k+1 \end{cases}$$

Since $\sin 0 = 0$ and $\cos 0 = 1$, the Taylor polynomials of $\sin x$ are given by

$$p_0(\sin x; 0) \;=\; 0,\;\; p_1(\sin x; 0) \;= p_2(\sin x; 0) \;=\; x,\;\; p_3(\sin x; 0) \;=\; p_4(\sin x; 0) \;=\; x - \frac{x^3}{6}, \ldots$$

More generally, for any positive integer k,

$$p_{2k-1}(\sin x; 0) \;=\; p_{2k}(\sin x; 0) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots - (-1)^k \frac{x^{2k-1}}{(2k-1)!}. \tag{10.1.7}$$

(**2**): Put

$$f(x) \;=\; \log x.$$

This function is not defined at 0, so we need to choose another point to evaluate the derivatives, and the easiest one is

$$a \;=\; 1.$$

We have

$$f'(x) \;=\; \frac{1}{x},\; f''(x) \;=\; -\frac{1}{x^2},\; f'''(x) \;=\; \frac{2!}{x^3}, \ldots$$

By induction, we have for any $n \geq 1$,

$$f^{(n)}(x) \;=\; (-1)^{n+1}\frac{(n-1)!}{x^n}.$$

So the nth Taylor coefficient is

$$\frac{f^{(n)}(1)}{n!} \;=\; (-1)^{n+1}\frac{1}{n},$$

where we have used the simple fact that $n!$ is n times $(n-1)!$. Consequently, since $\log 1 = 0$, the nth Taylor polynomial of $\log x$ is given by

$$p_n(\log x; 1) \;=\; x - \frac{x^2}{2} + \ldots + (-1)^{n+1}\frac{x^n}{n}. \tag{10.1.8}$$

(**3**): Consider

$$g(x) \;=\; \frac{1}{x}.$$

One has, for every $n \geq 0$,

$$g^{(n)}(x) \;=\; f^{(n+1)}(x),$$

where $f(x)$ is $\log x$. Thus for any $a > 0$,

$$\frac{g^{(n)}(1)}{n!} \;=\; (n+1)\frac{f^{(n+1)}(a)}{(n+1)!}. \tag{10.1.9}$$

As a consequence the Taylor polynomials of g at $a = 1$ are determinable from those of f. Let us make this idea precise.

Lemma 10.1.10 *Let f be a which is n times differentiable around a, with*

$$p_n(f(x);a)) = a_0 + a_1(x-a) + \ldots + a_n(x-a)^n.$$

Then

$$p_{n-1}(f'(x);a) = a_1 + 2a_2 x + \ldots + na_n(x-a)^{n-1}.$$

Moreover, if ϕ is a primitive of f around a,

$$p_n(\phi(x);a) = \phi(a) + a_0(x-a) + a_1\frac{(x-a)^2}{2} + \ldots + a_{n-1}\frac{(x-a)^n}{n}.$$

The proof is immediate from the definition of Taylor polynomials.

For a general f, even for such a simple function like $\frac{1}{1+x^2}$, it is painful to work out the Taylor polynomials from scratch. One needs a better way to find them, and this will be accomplished in the next section.

10.2 Approximation to order n

Definition 10.2.1 *Let f, g be n times differentiable functions at a. We will say that f and g* **agree up to order** *n at a iff we have*

$$\lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^n} = 0.$$

If g is a polynomial agreeing with f (or equalling f, as some would say) up to order n, then we would call g a **polynomial approximation of** $f(x)$ **to oder** n **at** $x = a$. The immediate question which arises is whether the nth Taylor polynomial of f is a polynomial approximation to order n. The answer turns out to be **Yes**, but even more importantly, the Taylor polynomial is the only one which has this property. Here is the complete statement!

Proposition 10.2.2 *Let f be n times differentiable at a. Then*

(i) $p_n(f(x);a)$ is a polynomial approximation of f to order n;

(ii) If $q(x)$ is any polynomial in $(x-a)$ of degree $\leq n$ which agrees with f up to order n, then $q(x) = p_n(f(x);a)$.

Proof. (**i**): Put

(10.2.3) $$g(x) = p_{n-1}(f(x);a) \quad \text{and} \quad h(x) = (x-a)^n.$$

Then by definition,

$$p_n(f(x);a) = g(x) + \frac{f^{(n)}(a)}{n!}h(x).$$

Hence
$$\frac{f(x) - p_n(f(x);a)}{(x-a)^n} \;=\; \frac{f(x) - g(x)}{h(x)} - \frac{f^{(n)}(a)}{n!}.$$
So it suffices to prove the following:

$$(10.2.4) \qquad \lim_{x \to a} \frac{f(x) - g(x)}{h(x)} \;=\; \frac{f^{(n)}(a)}{n!}.$$

Applying Lemma 10.1.2, we get

$$(10.2.5) \qquad g^{(j)}(a) \;=\; f^{(j)}(a) \;\forall\, j < n.$$

Since g is a polynomial of degree $\leq n-1$, its $(n-1)$th derivative is a constant; so

$$(10.2.6) \qquad g^{(n-1)}(x) \;=\; g^{(n-1)}(a).$$

Also,

$$(10.2.7) \qquad h^{(j)}(x) \;=\; \frac{n!(x-a)^{n-j}}{(n-j)!}.$$

It follows from (10.2.5) and (10.2.7) that for every $j < n-1$,

$$(10.2.8) \qquad \lim_{x \to a} f^{(j)}(x) - g^{(j)}(x) \;=\; \frac{f^{(j)}(a) - g^{(j)}(a)}{h^{(j)}(a)} \;=\; 0$$

and
$$\lim_{x \to a} h^{(j)}(x) \;=\; h^{(j)}(a) \;=\; 0.$$

On the other hand, by (10.2.6) and (10.2.7),

$$(10.2.9) \qquad \lim_{x \to a} \frac{f^{(n-1)}(x) - g^{(n-1)}(x)}{h^{(n-1)}(x)} \;=\; \lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{n!(x-a)} \;=\; \frac{f^{(n)}(a)}{n!}.$$

In view of (10.2.8) and (10.2.9), we may apply L'Hopital's rule (see the Appendix to chapter 9) and deduce (10.2.4), which also proves part (i) of the Proposition.

(**ii**): By hypothesis, $q(x)$ approximates $f(x)$ to order n at a. By part (i), the Taylor polynomial $p_n(f(x);a)$ does the same thing. It follows, since the limit of a sum is the sum of the limits, that $q(x)$ and $p_n(x)$ agree up to order n. Put

$$u(x) \;=\; p_n(f(x);a) - q(x),$$

which is a polynomial of degree $\leq n$ and satisfies

$$\lim_{x \to a} \frac{u(x)}{(x-a)^n} \;=\; 0.$$

This implies in particular that

$$\lim_{x\to a} \frac{u(x)}{(x-a)^j} \;=\; 0\ \forall\, j \le n. \tag{10.2.10}$$

On the other hand, applying the *Euclidean algorithm* repeatedly, relative to the divisor $(x-a)$, we can find, as we did in the chapter on*partial fractions*, numbers $c_0, \ldots, c_n$ such that

$$u(x) \;=\; c_0 + c_1(x-a) + \ldots + c_n(x-a)^n.$$

It is then immediate that for any $j \le n$,

$$\lim_{x\to a} \frac{u(x)}{(x-a)^j} \;=\; c_j.$$

In view of (10.2.10), this means that every coefficient c_j is zero. Thus the polynomial $u(x)$ is identically zero.

□

Now let us apply this to compute the Taylor polynomials of

$$\phi(x) \;=\; \arctan x \tag{10.2.11}$$

at $a = 0$, where ϕ takes the value 0. (You may try as an educational exercise to compute directly with $\phi(x)$, and you will learn why this Proposition is helpful.)

Recall that ϕ is a primitive of

$$f(x) \;=\; \frac{1}{1+x^2} \tag{10.2.12}$$

for all x in the domain of $\arctan x$, namely the open interval $(-\pi/2, \pi/2)$. Also, f is infinitely differentiable everywhere.

We will compute the Taylor polynomials of $f(x)$ at 0 by the following trick. For each $n \ge 1$, look at the polynomial

$$g_n(x) \;=\; 1 - x^2 + \ldots + (-1)^n x^{2n}.$$

It is a *geometric sum* and so we can reexpress it as

$$g_n(x) \;=\; \frac{1 + x^{2(n+1)}}{1+x^2} \;=\; f(x) + \frac{x^{2n+2}}{1+x^2}.$$

Consequently, for $r = 2n, 2n+1$

$$\lim_{x\to 0} \frac{f(x) - g_n(x)}{x^r} \;=\; \lim_{x\to a} \frac{x^{2n+2-r}}{1+x^2} \;=\; 0.$$

Thus $g_n(x)$ approximates $f(x)$ to order $2n$ and $2n+1$, so it must, by the Proposition above, equal the Taylor polynomials $p_{2n}(f(x);0)$ and $p_{2n+1}(f(x);0)$. Thus for any $n \ge 0$,

$$p_{2n}(\frac{1}{x};0) \;=\; p_{2n+1}(\frac{1}{x};0) \;=\; 1 - x^2 + \ldots + (-1)^n x^{2n}. \tag{10.2.13}$$

Applying Lemma 10.1.10, we then deduce that for all $n \geq 0$,
(10.2.14)

$$p_{2n+1}(\arctan x; 0) \;=\; p_{2n+2}(\arctan x; 0) \;=\; p_0(\arctan x; 0) + x - \frac{x^3}{3} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

10.3 Taylor's Remainder Formula

Once one has looked at the Taylor polynomials $p_n(f; a)$ of a sufficiently differentiable function f at a point a, the natural question which arises immediately is how close an approximation to f does one get this way. To be precise, define the n**th remainder of** f **at** a to be

(10.3.1)
$$R_n(f(x), a) \;=\; f(x) - p_n(f(x); a).$$

A very precise answer to this question wa supplied by Taylor. Here it is!

Theorem 10.3.2 *Let $n \geq 0$, $a < x \in \mathbb{R}$, and f an $(n+1)$-times differentiable function on an open interval containing $[a, x]$. Then we have the following:*

(a)

$$R_n(f(x); a) \;=\; \frac{f^{(n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c in (a, x).

(b) If $f^{(n+1)}$ is integrable on $[a, x]$, then

$$R_n(f(x); a) \;=\; \frac{1}{n!} \int_a^x f^{(n+1)}(u)(x-u)^n du.$$

Corollary 10.3.3 *Let f be $(n+1)$-times differentiable on $[a, x]$. Suppose there are numbers m, M such that*

$$m \;\leq\; f^{(n+1)}(u) \;\leq\; M$$

for **all** *u in $[a, x]$. Then we have*

(i)
$$m\frac{(x-a)^{n+1}}{(n+1)!} \;\leq\; R_n(f)x); a) \;\leq\; M\frac{(x-a)^{n+1}}{(n+1)!}.$$

In particular, if $C = \max\{|m|, |M|\}$,

(ii)
$$|R_n(f)x); a)| \;\leq\; C\frac{(x-a)^{n+1}}{(n+1)!}.$$

Completely analogous assertions hold when $x < a$, in which case one should replace $[a, x]$, everywhere in the Theorem and Corollary with $[x, a]$, resp. $a - x$.

Let us first look at the **example** of the exponential function. We know that

$$f(u) = \exp(u)$$

is infinitely differentiable on all of $\mathbb{R}$ with $f'(u) = f(u)$. Moreover, since e^u is an increasing function with $e^0 = 1$, we get, for $x > 0$, $u \in [1, x]$ and $n \geq 0$,

$$1 \leq f^{(n+1)}(u) \leq e^x.$$

Consequently, by corollary 10.3.3, we have

$$\frac{x^{n+1}}{(n+1)!} \leq R_n(e^x; 0) \leq e^x \frac{x^{(n+1)}}{(n+1)!}. \tag{10.3.4}$$

Suppose we want to evaluate e to within an error of 10^{-4}. Then what we have to do is the following. Putting $x = 1$ in (8.3.4), and remembering the crude estimate that e is less than 3, we obtain

$$\frac{1}{(n+1)!} \leq R_n = R_n(e; 0) \leq \frac{3}{(n+1)!}. \tag{10.3.5}$$

Find the smallest n for which

$$\frac{3}{(n+1)!} < 10^{-4}.$$

Direct computation shows that

$$\frac{3}{7!} = \frac{3}{5040} = \frac{1}{1680} > 10^{-4}$$

and

$$\frac{3}{8!} = \frac{3}{40320} = \frac{1}{13440} < 10^{-4}.$$

So we take $n = 7$, and the error will be less than 10^{-4} if we approximate e by

$$p_7(e; 0) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!},$$

which is

$$\frac{13700}{5040} = 2.7182539\ldots$$

The first four places after the decimal point are correct, as they should be. But at the fifth place the digit should be 8 instead of 5, and to get that one has to go to the n (namely 8) which makes R_n less than 10^{-5}.

The remainder formula applied to the functions $\sin x$ and $\cos x$ yields very similar estimates for the remainder. To be precise, we use the fact that the Taylor polynomials of $\sin x$, resp. $\cos x$, at $x = 0$, have only odd, resp. even, degree terms, and obtain the following:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + R_{2m+1}(\sin x; 0), \tag{10.3.5}$$

with

$$|R_{2m+1}(\sin x; 0)| \leq \frac{|x|^{2m+3}}{(2m+3)!};$$

and

$$\text{(10.3.6)} \qquad \cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + R_{2m}(\cos x; 0),$$

with

$$|R_{2m+1}(\cos x; 0)| \leq \frac{|x|^{2m+2}}{(2m+2)!}.$$

It is a simple exercise to approximate numbers like $\sin 1$ or $\cos(1/2)$ to any number of decimel places.

Taylor's formula is not very useful, however, for estimating the remainders of functions f for which it is hard to get a nice expression for $f^{(n+1)}(u)$. A very **important example** illustrating this phenomenon is the function

$$f(x) \;=\; \arctan x, \;\; x \in (-\pi/2, \pi/2).$$

So what does one do? After some reflection, one remembers the method by which one found the Taylor polynomials of this functions. Luckily, this method also leads to a good estimate for the remainder. Let us see how.

Recall that

$$\frac{d}{dx}(\arctan x) \;=\; \frac{1}{1+x^2},$$

and that

$$\frac{1-(-1)^m x^{2m+2}}{1+x^2} \;=\; 1 - x^2 + x^4 - \ldots + (-1)^m x^{2m}.$$

The second formula can be rewritten as

$$\frac{1}{1+x^2} \;=\; 1 - x^2 + x^4 - \ldots + (-1)^m \frac{x^{2m+2}}{1+x^2}.$$

Integrating this expression and using the fact that $\arctan 0 = 0$, we get by the fundamental theorem of Calculus,

$$\text{(10.3.8}-i) \qquad \arctan x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + R_{2m+1}(\arctan x; 0),$$

where

$$R_{2m+1}(\arctan x; 0) \;=\; (-1)^m \int_0^x \frac{u^{2m+3}}{1+u^2}.$$

I am using here the fact that we have already seen that the polynomial $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1}$ is the Taylor polynomial by the criterion given by part (ii) of Proposition 10.2.2. Suppose $x > 0$. Then

$$\frac{u^{2m+3}}{1+u^2} \;\leq\; u^{2m+3} \;\forall\, u \in [0, x],$$

in fact with equality only for $u = 0$, and hence

$$|R_{2m+1}(\arctan x; 0)| \;\leq\; \int_0^x u^{2m+3} du.$$

Since the integral of u^{2m+3} is $u^{2m+4}/(2m+4)$, we get the desired bound

$$(10.3.8-ii) \qquad |R_{2m+1}(\arctan x; 0)| \;<\; \frac{x^{2m+3}}{2m+3}.$$

Note that for any fixed $x > 0$, this expression goes to 0 as m goes to ∞. By taking $x = 1$, and letting $m \to \infty$, one gets the **Leibniz formula**:

$$\frac{\pi}{4} \;=\; \arctan 1 \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^m \frac{1}{2m+1} + \ldots$$

This is no doubt a beautiful formula, but it is not quite useful for computations, because $1/m$ goes to 0 rather slowly, at least compared to $1/n!$, which is what one had for the exponential or the sine function. The silver lining is that while (10.3.8-ii) is not decreasing fast for $x = 1$, it converges faster when x is small. To exploit this, one appeals to the **addition theorem for the arctangent**, namely

$$(10.3.9) \qquad \arctan x + \arctan y \;=\; \arctan\left(\frac{x+y}{1-xy}\right),$$

which follows by applying the inverse function arctan to the addition theorem for the tangent function (with $x = \tan u, y = \tan v$):

$$\tan(u+v) \;=\; \frac{\tan u + \tan v}{1 - \tan u \tan v}.$$

¿From this one can derive, for example the following identities:

$$\frac{\pi}{4} \;=\; \arctan 1 \;=\; \arctan(1/2) + \arctan(1/3)$$

and

$$\frac{\pi}{4} \;=\; \arctan 1 \;=\; 4\arctan(1/5) - \arctan(1/239).$$

The second formula, proved by Machin in 1706, can be used to find the first five or six decimal places of π very fast. (Of course Mathematica or Maple can spew out the first 10,000 digits in a few seconds, but the methods used there are very sophisticated and appeal to formulas involving elliptic functions.)

Proof of Theorem 10.3.2, For every u in $[a, x]$, we have

$$(10.3.10) \qquad f(x) \;=\; p_n(f(x); u) + R_n(f(x); u),$$

where

$$(10.3.11) \qquad p_n(f(x);u) \;=\; f(u) + f'(u)(x-u) + \frac{f''(u)}{2!}(x-u)^2 + \ldots + \frac{f^{(n)}(u)}{n!}(x-u)^n.$$

Note that
(10.3.12)

$$\frac{d}{du}p_n(f(x);u) \;=\; f'(u)+(-f'(u)+f''(u)(x-u))+\left(-f''(u)(x-u)+\frac{f^{(3)}(u)}{2!}(x-u)^2\right)+\ldots+\Big(-\frac{-f^{(n)}}{(n-}$$

Differentiating both sides of (10.3.10) and making use of (10.3.12), we obtain, for every u in $[a,x]$,

$$(10.3.13) \qquad 0 \;=\; \frac{f^{(n+1)}(u)}{n!}(x-u)^n + \frac{d}{du}R_n(f(x);u).$$

The function $R_n(f(x);u)$ is continuous on $[a,x]$ and differentiable, because $f(x)$ and $p_n(f(x);u)$ are, on (a,x). Of course the polynomial function $\phi(u) = (x-u)^{n+1}$ has the same properties. So we may apply the Cauchy Mean Value Theorem (see the Appendix to chapter 9) to $R_n(f(x);u)$ and $\phi(u)$ and get a number c in (a,x) such that

$$(10.3.15) \qquad \frac{d\phi}{du}(c)(R_n(f(x);x) - R_n(f(x);a)) \;=\; \frac{d}{du}R_n(f(x);u)(c)(\phi(x)-\phi(a)).$$

By (10.3.13),

$$(10.3.16) \qquad \frac{d}{du}R_n(f(x);u)(c) \;=\; -\frac{f^{(n+1)}(c)}{n!}(x-c)^n.$$

And

$$(10.3.17) \qquad \frac{d\phi}{du}(c) \;=\; -(n+1)(x-c)^n.$$

Combining (10.3.15), (10.3.16) and (10.3.17), cancelling $-(x-u)^n$, and dividing by $(n+1)$, we obtain the formula (i). (This particular form of the remainder was in fact derived by Lagrange.)

Now suppose $f^{(n+1)}$ is integrable on $[a,x]$. Then applying the fundamental theorem of Calculus, and remembering that $R_n(f(x);x) = 0$, we get

$$-R_n(f(x);a) \;=\; \int_a^x \left(\frac{d}{du}R_n(f(x);u)\right) du,$$

whose right hand side expression is, by (10.3.13),

$$-\int_a^x \frac{f^{(n+1)}(u)}{n!}(x-u)^n du.$$

Hence we get (ii).

□

10.4 The irrationality of e

Now let us prove that e is irrational. It is even transcendental, but that is much harder to prove.

Suppose $e = p/q$ for some positive integers p, q. Choose an integer $n > 3$ which is greater than q. Using (10.3.4) and (10.3.5), we get

$$e = \frac{p}{q} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n,$$

with

$$0 < \frac{3}{(n+1)!}.$$

Multiplying throughout by $n!$, we get

$$n!\frac{p}{q} = n! + n! + \frac{n!}{2!} + \ldots + \frac{n!}{n!} + n!R_n.$$

But since $n > q$, $\frac{n!}{q}$ is an integer; so is $\frac{n!}{j!}$ for any positive integer $j \leq n$. This implies that $n!R_n$ is an integer. But

$$0 < n!R_n < \frac{n!3}{(n+1)!} = \frac{3}{n+1},$$

and this gives a contradiction because $n > 3$, implying that $3/(n+1)$ is < 1.

Hence e must be irrational!

□